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Ocean Turbulence II: one–point closure model
Momentum, heat and salt vertical diffusivities
in the presence of shear

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Abstract

We develop and test a 1–point closure turbulence model with the following features.

1) we include the salinity field and derive the expression for the vertical turbulent diffusivities of momentum K_m, heat K_h and salt K_s as a function of two stability parameters: the Richardson number Ri (stratification vs. shear) and the Turner number R_ρ (salinity gradient vs. temperature gradient).

2) to describe turbulent mixing below the mixed layer (ML), all previous models have adopted three adjustable "background diffusivities" for momentum, heat and salt We propose a model that avoids such adjustable diffusivities *We assume that below the ML, the three diffusivities have the same functional dependence on Ri and R_ρ as derived from the turbulence model* However, in order to compute Ri below the ML, we use data of vertical shear due to wave–breaking measured by Gargett et al. (1981). The procedure frees the model from adjustable background diffusivities and indeed we employ the same model throughout the entire vertical extent of the ocean.

3) in the local model, the turbulent diffusivities $K_{m,h,s}$ are given as analytical functions of Ri and R_ρ

5) the model is used in an O–GCM and several results are presented to exhibit the effect of double diffusion processes.

6) the code is available upon request.

I. Introduction

For sake of completeness, we recall that the O–GCM solve the dynamic equations for the mean velocity U_i, mean temperature T and mean salinity S:

$$\frac{\partial}{\partial t}U_i + \frac{\partial}{\partial x_j}(U_iU_j + \overline{u_i''u_j''}) = \dots \tag{1a}$$

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_i}(U_iT + \overline{u_i''T''}) = \dots \tag{1b}$$

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x_i}(U_iS + \overline{u_i''s''}) = \dots \tag{1c}$$

The velocity, temperature and salinity fields have also fluctuating components u_i'', T'' and s'' which produce the correlations $\overline{u_i''u_j''}$ (Reynolds stresses), $\overline{u_i''T''}$(heat fluxes) and $\overline{u_i''s''}$(salinity fluxes). The challenge then is to construct such correlations so as to solve Eqs (1a–c). To fix the ideas, we further write:

$$\overline{u_i''u_j''} = -K_m\Sigma_{ij} \tag{1d}$$

$$\overline{u_i''T''} = -K_h\frac{\partial T}{\partial x_i} \tag{1e}$$

$$\overline{u_i''s''} = -K_s\frac{\partial S}{\partial x_i} \tag{1f}$$

where $\Sigma_{ij}=\frac{1}{2}(U_{i,j}+U_{j,i})$ is the mean shear. The $K_{m,h,s}$ are the *turbulent diffusivities* for momentum, heat and salt. As discussed in paper I, they have the general functional form.

$$K_{m,h,s} = 2\,\frac{K^2}{\epsilon}S_{m,h,s} \tag{1g}$$

where K and ϵ are the turbulent kinetic energy and its rate of dissipation which in principle are given by two dynamic equations (the K–ϵ model). The *dimensionless structure functions* $S_{m,h,s}$ must differ from one another so that:

$$K_m \neq K_h \neq K_s \tag{1h}$$

In general we can write.

$$S_{m,h,s} = S_{m,h,s}(\nabla U,\ \alpha_T\nabla T,\ \alpha_s\nabla S) \tag{1i}$$

where α_T and α_s are the volume expansion coefficients $\alpha_T=-\rho^{-1}\partial\rho/\partial T$ and $\alpha_s=\rho^{-1}\partial\rho/\partial S$ and where the shear $\nabla\mathbf{U}$ can be generated either by external sources like in the mixed layer ML or by internal wave–breaking processes below the ML. If one introduces the Turner number

R_ρ and the Richardson number Ri:

$$R_\rho = g\alpha_S\frac{\partial S}{\partial z}\left(g\alpha_T\frac{\partial T}{\partial z}\right)^{-1}, \quad \mathrm{Ri} = N_h^2/N_u^2 \tag{2a}$$

where

$$N_h^2 = g\alpha_T\frac{\partial T}{\partial z}, \quad N_u^2 = 2(\Sigma_{ij}\Sigma_{ij})$$
$$N^2 = -\frac{g}{\rho}\frac{\partial\rho}{\partial z} = g\alpha_T\frac{\partial T}{\partial z} - g\alpha_S\frac{\partial S}{\partial z} = N_h^2(1-R_\rho) \tag{2b}$$

we can rewrite (1i) more concisely as

$$S_{m,h,s} = S_{m,h,s}(R_\rho, \mathrm{Ri}) \tag{2c}$$

(Clearly, we could have also defined Ri in terms of N^2 rather than just the thermal gradient. We have chosen the latter for reasons of presentation of the results). One must distinguish the following four cases:

SF (*salt fingers, salty–warm over fresh–cold*):

$$\frac{\partial S}{\partial z}>0, \quad \frac{\partial T}{\partial z}>0,$$
$$R_\rho>0, \quad \mathrm{Ri}>0$$
$$R_\rho<1 \text{ Stable, } N^2>0, \quad R_\rho>1 \text{ Unstable, } N^2<0 \tag{2d}$$

DC (*diffusive convection, fresh–cold over salty–warm*):

$$\frac{\partial S}{\partial z}<0, \quad \frac{\partial T}{\partial z}<0,$$
$$R_\rho>0, \quad \mathrm{Ri}<0$$
$$R_\rho<1 \text{ Unstable, } N^2<0, \quad R_\rho>1 \text{ Stable, } N^2>0 \tag{2e}$$

DS (*doubly stable, fresh–warm over salty–cold*)

$$\frac{\partial S}{\partial z}<0, \quad \frac{\partial T}{\partial z}>0,$$
$$R_\rho<0, \quad \mathrm{Ri}>0, \quad N^2>0, \quad \text{Stable} \tag{2f}$$

DU (*doubly unstable, salty–cold over fresh–warm*):

$$\frac{\partial S}{\partial z}>0, \quad \frac{\partial T}{\partial z}<0,$$
$$R_\rho<0, \quad \mathrm{Ri}<0, \quad N^2<0, \quad \text{Unstable} \tag{2g}$$

The stability/instability is predicated on the Brunt–Vaisala frequency N with $N^2>0$ (stable) and $N^2<0$ (unstable).

The general problem is to construct (2c) so as to encompass all four cases (2d–g)

First, there is ample evidence from laboratory and oceanic field data that show that K_h is different from K_s. In the SF case, the ratio $K_s/K_h>1$ (Hamilton et al., 1989) while in the DC case, $K_h/K_s>1$ (Kelley, 1984). Schmitt (1981) has also shown that the observed T–S relationship is not consistent with $K_h=K_s$. For a discussion and review of the importance of these processes and their extent in different parts of the ocean, see Turner (1967, 1973, 1985), Schmitt (1994) and Zhang et al. (1998). In spite of this evidence, almost all O–GCM still assume

$$K_s=K_h \tag{3a}$$

Recently, attempts have been made to overcome (3a) but the task is not easy The main difficulty is that in the absence of a model capable of encompassing all four cases, SF, DC, DS and DU, the only alternative is to employ laboratory and ocean data to build the functional form of the diffusivities to be used in an O–GCM. This is the approach employed by Large et al. (1994), Zhang et al. (1998, ZSH) and Merryfield et al. (1999, MHG) who used relations by Schmitt (1981) and Kelley (1984, 1990), among others.

There is, however, an internal limitation to such a procedure since the available data refer to SF and DC but not to DS and DU which are also important (Duffy and Caldera, 1999) Thus, away from the regions where SF and DC are active, the above authors take

$$K_{m,h,s}(\mathrm{DS},\ \mathrm{DU})=0 \tag{3b}$$

or, more precisely, they use a background diffusivity which is chosen primarily on grounds of numerical stability but whose physical origin must be an internal–wave breaking phenomenon This is clearly not a satisfactory situation especially in view of the fact that Since the studies by ZHS and MHG have shown the importance of double diffusion, the above procedure is certainly far better than (3a) but still not fully satisfactory. The goal of this paper is to consider the same problem but with a different methodology.

We develop *a turbulence model to compute the three diffusivities for momentum, heat and salt* and construct the functions (1g) and (1i) for the four processes SF, DC, DS and DU in the presence of an arbitrary shear. The inclusion of shear is quite relevant since is

known to hamper the SF mechanism (Linden, 1971, 1974a, b; Kunze, 1990) and yet, the above procedures do not account for shear since they expressed K_h and K_s in terms of only one stability parameter R_ρ, rather than R_ρ and the Richardson number Ri

We present three models: 1) K and ϵ are solutions of two dynamical equations (K–ϵ model), 2) only one of them satisfies a differential equation while the other is taken to be the local limit of its dynamic equation and 3) both K and ϵ are taken as the local limit of their respective dynamic equations. As we shall show, in model 3) all the relations are algebraic and one must solve a cubic equation. The numerical results correspond to 3).

The structure of the paper is as follows. In II–VII we derive the general non–local, dynamic equations for the mean fields as well as the turbulent variables. In VIII we derive the analytic expressions for the turbulent diffusivities with only two non–local variables, the turbulent kinetic energy K and its rate of dissipation, ϵ In IX–X we study the case of double diffusion without shear and show that the predictions of the model are in agreement with several laboratory and ocean data. In XI we give the complete analytic solution for the local model: we derive the algebraic expressions for the momentum, heat and salt diffusivities in the presence of arbitrary shear. In XII, we discuss the time scales. In XIII, we display several solutions of the model, specifically we plot the diffusivities K's and their ratios as a function of the two stability parameters, the Richardson number and the Turner number. In XIV–XVI we present the results of an O–GCM with the above model where we use the same turbulence model below the mixed layer where the shear is no longer due to the external wind forcing but to a wave breaking mechanism. In XVII we present some concluding remarks.

II. Continuity equation

Following the formalism presented elsewhere (Canuto, 1997), the total velocity, density and pressure fields are split into mean and fluctuating parts as follows:

$$u_i = U_i + u''_i, \quad \rho=\bar{\rho}+\rho', \; p=P+p', \quad \overline{p'}=\overline{\rho'}=0 \tag{5a}$$

the Reynolds average $<u_i'>=0$. The relation between the two is discussed in Canuto (1997a). Using the equation for the density ρ

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0, \quad \frac{d\rho}{dt} + \rho\frac{\partial}{\partial x_i}u_i = 0, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i} \tag{5c}$$

we obtain, upon mass averaging,

$$\frac{D}{Dt}\bar{\rho} + \bar{\rho}\frac{\partial}{\partial x_i}U_i = 0\ , \qquad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_i\frac{\partial}{\partial x_i} \tag{5d}$$

III. Velocity Field. Mean and Turbulent Variables

Consider the Navier–Stokes equations

$$\frac{\partial}{\partial t}\rho u_i + \frac{\partial}{\partial x_j}\rho u_i u_j = -\frac{\partial p}{\partial x_i} - \rho g_i + \frac{\partial}{\partial x_j}\sigma_{ij} \tag{5e}$$

where σ_{ij} is the viscous stress tensor (ν is the kinematic viscosity)

$$\sigma_{ij} = \nu\rho\left(\frac{\partial}{\partial x_j}u_i + \frac{\partial}{\partial x_i}u_j\right) - \frac{2}{3}\nu\rho\delta_{ij}\frac{\partial}{\partial x_k}u_k \tag{5f}$$

Mass averaging (5e), we obtain the dynamic equation for the large scale flow **U**,

$$\bar{\rho}\frac{D}{Dt}U_i = -\frac{\partial}{\partial x_j}(P\delta_{ij} + \tau_{ij}) - \bar{\rho}g_i \tag{5g}$$

where τ_{ij} are the turbulent Reynolds stresses

$$\tau_{ij} \equiv \overline{\rho u_i'' u_j''} = \bar{\rho}R_{ij} \tag{5h}$$

The kinetic energy of the large scale field

$$K_u = \tfrac{1}{2}U_i U_i \tag{5i}$$

satisfies the equation ($a_{,i} = \partial a/\partial x_i$; $a_{ij,k} \equiv \partial a_{ij}/\partial x_k$)

$$\bar{\rho}\frac{DK_u}{Dt} = -U_i(P_{,i} + \tau_{ij,j} + \bar{\rho}g_i) \tag{5j}$$

The Reynolds stresses R_{ij} satisfy the non–local dynamic equation (Canuto 1997):

$$\bar{\rho}\left(\frac{D}{Dt}R_{ij} + D_{ij}\right) = A_{ij} + B_{ij} - \pi_{ij} + \delta_{ij}PD - \bar{\rho}\epsilon_{ij} \tag{6a}$$

where the non–local term D_{ij} represents the flux of Reynolds stresses R_{ijk}:

$$D_{ij} = \bar{\rho}^{-1}\frac{\partial}{\partial x_k}\left[\bar{\rho}R_{ijk} + \frac{2}{3}\delta_{ij}\overline{p' u_k''} - \overline{\sigma_{ik}u_j} - \overline{\sigma_{jk}u_i}\right] \tag{6b}$$

$$R_{ijk} \equiv \bar{\rho}^{-1}\tau_{ijk} = \bar{\rho}^{-1}\,\overline{\rho u_i'' u_j'' u_k''} \tag{6c}$$

In Eq.(6a), the source term due to shear is represented by:

$$-A_{ij} \equiv \bar{\rho}[R_{ik}U_{j,k} + R_{jk}U_{i,k}] \tag{6d}$$

while the source (sink) term due to stratification is represented by

$$\bar{\rho}B_{ij} = (\overline{\rho' u''_j}\delta_{ik} + \overline{\rho' u''_i}\delta_{jk})P_{,k} \tag{6e}$$

The fluctuating pressure p' gives rise to the pressure–velocity correlation

$$\Pi_{ij} = \overline{u''_i p'_{,j}} + \overline{u''_j p'_{,i}}, \qquad \pi_{ij} \equiv \Pi_{ij} - \frac{1}{3}\delta_{ij}\Pi_{kk} \tag{6f}$$

Finally, compressibility introduces a pressure–dilatation term

$$PD = \frac{2}{3}\,\overline{p' u''_{i,i}} \equiv \frac{2}{3}\,\overline{p'd} \tag{6g}$$

where $d = u''_{i,i}$ is the "dilatation", while ϵ_{ij} is the dissipation tensor which we assume diagonal for its largest contribution originates in the small scales region:

$$\epsilon_{ij} = \frac{2}{3}\bar{\rho}\epsilon\delta_{ij} \tag{6h}$$

Below, we present the dynamic equation for ϵ The trace of (6a) yields the equation for the turbulent kinetic energy K,

$$K \equiv \tfrac{1}{2}\bar{\rho}^{-1}\,\overline{\rho u''_i u''_i} = \tfrac{1}{2}R_{ii} \tag{7a}$$

$$\frac{D}{Dt}K + D_f = -R_{ij}U_{i,j} + \bar{\rho}^{-2}\,\overline{\rho' u''_i}\,P_{,i} + \bar{\rho}^{-1}\,\overline{p'd} - \epsilon \tag{7b}$$

where $D_f(K)$ is the non–local transport of K:

$$D_f \equiv \bar{\rho}^{-1}\frac{\partial}{\partial x_i}(\tfrac{1}{2}\bar{\rho}\,R_{kki} + \overline{p' u''_i} - \overline{\sigma_{ij} u_j}) \tag{7c}$$

IV. Concentration Equations

Consider a model with two fluids of density ρc and $\rho(1-c)$, where c is the concentration and ρ the total density of the fluid which satisfies (5c). The equation satisfied by ρc is given by (no external sources)

$$\frac{\partial}{\partial t}\rho c + \frac{\partial}{\partial x_i}(\rho c u_i) = (\rho J_i)_{,i} \tag{8a}$$

or alternatively,

$$\rho\frac{dc}{dt} = (\rho J_i)_{,i} \tag{8b}$$

where J_i is the diffusion flux

$$J_i = \chi_c\frac{\partial c}{\partial x_i} \tag{8c}$$

where χ_c is the molecular diffusivity of the c–field. Mass averaging, we derive

$$\overline{\rho c} = \bar{\rho}C, \qquad \overline{\rho c u_i} = \bar{\rho} C U_i + \bar{\rho}\Phi_i \tag{9a}$$

where C is mean concentration and Φ_i is *turbulent concentration flux*

$$C \equiv \tilde{c}, \qquad \overline{\rho u_i'' c''} = \bar{\rho}\Phi_i \tag{9b}$$

Taking the mass average of Eq. (8b), we obtain the equation for the mean concentration C:

$$\bar{\rho}\frac{DC}{Dt} = \frac{\partial}{\partial x_i}[\bar{\rho}\chi_c\frac{\partial C}{\partial x_i} - \bar{\rho}\Phi_i] \tag{10a}$$

To provide the turbulent flux Φ_i, we need a turbulence model.

V. Temperature field

We begin with the equation for the entropy S (Landau and Lifshitz 1987),

$$\rho T\frac{dS}{dt} = -\frac{\partial}{\partial x_i}(q_i + \mu J_i) + J_i\frac{\partial \mu}{\partial x_i} + \sigma_{ij}\frac{\partial u_i}{\partial x_j} \tag{11}$$

where μ is the chemical potential and

$$q_i = F_i^r - J_i(\mu - T\frac{\partial \mu}{\partial T}|_{p,c}) \equiv F_i^r - hJ_i \tag{12}$$

where F_i^r is the thermal flux. In the absence of diffusion, $q_i = F_i^r$ but here q_i depends also on the gradients of the concentration as well as on the gradient of μ. Using

$$\frac{dS}{dt} = \frac{\partial S}{\partial T}|_{c,p}\frac{dT}{dt} + \frac{\partial S}{\partial c}|_{T,p}\frac{dc}{dt} + \frac{\partial S}{\partial p}|_{c,T}\frac{dp}{dt} \tag{13}$$

and the relations:

$$T\frac{\partial S}{\partial T}|_{c,p} = c_p, \quad \frac{\partial S}{\partial c}|_{T,p} = -\frac{\partial \mu}{\partial T}|_{p,c}, \quad \rho^2\frac{\partial S}{\partial p}|_{c,T} = \frac{\partial \rho}{\partial T}|_{p,c} \tag{14}$$

one can transform Eq (11) as:

$$c_p[\frac{\partial}{\partial t}\rho T + \frac{\partial}{\partial x_i}(\rho u_i T)] = \omega\frac{dp}{dt} - \frac{\partial}{\partial x_i}F_i^r + \sigma_{ij}\frac{\partial u_i}{\partial x_j} + \rho\chi_c\frac{\partial c}{\partial x_k}\frac{\partial h}{\partial x_k} \tag{15a}$$

where the dimensionless function ω is defined by:

$$\omega = -T\rho^{-1}\frac{\partial \rho}{\partial T} \equiv \alpha_T T \tag{15b}$$

Next, we take the mass average of (15a). Making use of the results derived in Canuto (1997) and recalling that $\overline{\sigma_{ij}u_{j,i}} = \bar{\rho}\epsilon$, we obtain

$$\bar{\rho}c_p\frac{DT}{Dt} = -(F_i^c + \overline{F_i^r} - \omega\overline{p'u_i''})_{,i} + \omega\frac{DP}{Dt} + \omega\overline{u_i''}P_{,i} - \omega\overline{p'd} + \bar{\rho}\epsilon + \chi_c\,\overline{\rho c_{,k}h_{,k}}$$

$$(16a)$$

where F_i^c is the heat flux

$$F_i^c = c_p\overline{\rho T'' u_i''} = \bar{\rho}H_i \tag{16b}$$

The $\overline{u_i''}$ term in (16a) can be written using the second of (5b) and (22e) below. Eq (16a) is the generalized Bernoulli's equation with turbulence, diffusion and heat flux. When dealing with incompressible ocean turbulence, one can neglect the third–order term $\overline{p'u_i''}$ as being smaller than the second–order terms to which is summed; the last term can also be justifiably neglected as a kinematic term smaller than the remaining turbulence terms; the $\overline{p'd}$ term can also be neglected since $d=\partial u_i/\partial x_i=0$ in an incompressible flow; the term $\bar{\rho}\epsilon$ cannot in principle be neglected since it represents the energy gained by the temperature field that is lost by the kinetic energy because of friction; thus, its presence is a consequence of energy conservation Thus, we have:

$$\bar{\rho}c_p\frac{DT}{Dt} = -(F_i^c+\overline{F_i^r})_{,i} + \omega\left(\frac{D}{Dt} + \overline{u_i''}\frac{\partial}{\partial x_i}\right)P + \bar{\rho}\epsilon \tag{16c}$$

Since $D/Dt = \partial/\partial t + U_i\partial/\partial x_i$, one can probably neglect the $\overline{u_i''}$ term since it summed to the mean flow velocity U_i which is expected to be larger. This further reduces (16c) to:

$$\bar{\rho}c_p\frac{DT}{Dt} = -\frac{\partial}{\partial x_i}(F_i^c+\overline{F_i^r}) + \omega\frac{DP}{Dt} + \bar{\rho}\epsilon \tag{16d}$$

In all O–GCM, this equation is however further reduced to:

$$c_p\bar{\rho}\frac{DT}{Dt} = -\frac{\partial}{\partial x_i}(F_i^c+F_i^r) \tag{16d}$$

VI. Turbulence

As already discussed, we need to evaluate the following second–order moments (2c). The equation for the first of them has already been given by Eq.(6a). As for the correlation

$$\phi = \tfrac{1}{2}\overline{\rho c''^2} \equiv \bar{\rho}\Phi \tag{17}$$

we first note that using Eq.(8a), ρc^2 satisfies the dynamic equation ($J_{i,i}\equiv\partial\rho J_i/\partial x_i$)

$$\frac{\partial}{\partial t}\rho c^2 + \frac{\partial}{\partial x_i}(\rho u_i c^2) = 2cJ_{i,i} \tag{18a}$$

Taking into account the relations

$$\overline{\rho c^2} = \bar{\rho}\, C^2 + \overline{\rho c''^2} \tag{18b}$$

$$\overline{\rho u_i c^2} = \bar{\rho} U_i C^2 + U_i \overline{\rho c''^2} + \overline{\rho u_i'' c''^2} + 2C\overline{\rho u_i'' c''} \tag{18c}$$

the mass average of (18a) gives:

$$\frac{\partial}{\partial t}\phi + D_f(\phi) = -\phi U_{i,i} - \bar{\rho}\Phi_i C_{,i} + \overline{cJ}_{i,i} - CJ_{i,i} \tag{18d}$$

$$D_f(\phi) = \tfrac{1}{2}\bar{\rho}^{-1}\frac{\partial}{\partial x_i}(\overline{\rho u_i'' c''^2}) \tag{18e}$$

Introducing the new function Φ

$$\tfrac{1}{2}\overline{\rho c''^2} = \bar{\rho}\Phi \tag{19a}$$

Eq (18d) simplifies to:

$$\bar{\rho}[\frac{D}{Dt}\Phi + D_f(\Phi)] = -\bar{\rho}\Phi_i C_{,i} + \overline{cJ}_{i,i} - CJ_{i,i} \tag{19b}$$

where the non–local transport of Φ is given by

$$D_f(\Phi) = \tfrac{1}{2}\bar{\rho}^{-1}\frac{\partial}{\partial x_i}\overline{\rho u_i'' c''^2} \tag{19c}$$

The last two terms in (19b) will be evaluated as follows:

$$\overline{cJ}_{i,i} - CJ_{i,i} = CJ_{i,i} + \overline{c''J}_{i,i} - CJ_{i,i} = \overline{c''J}_{i,i} = \overline{c''J''}_{i,i} \tag{19d}$$

which, using Eq (8c) and considering the incompressibility of the flow, becomes (for ease of notation, we employ <..> instead of an overbar)

$$\bar{\rho}\chi_c < c''\frac{\partial^2 c''}{\partial x_j^2} > \tag{19e}$$

Using a mathematical identity, we also have

$$\bar{\rho}\chi_c < c''\frac{\partial^2 c''}{\partial x_j^2} > = -\bar{\rho}\chi_c < (\frac{\partial c''}{\partial x_i})^2 > + \bar{\rho}\chi_c \frac{\partial^2}{\partial x_i^2}(\tfrac{1}{2}\overline{c''^2}) \tag{19f}$$

The last term represents the viscous–diffusion of (17), which we may consider small while the first term is the viscous–dissipation which we cannot neglect. We recall that in the case of momentum, the dissipation ϵ, see Eq.(7b),

$$\epsilon_{ij} = 2\nu < \frac{\partial u_i''}{\partial x_k}\frac{\partial u_j''}{\partial x_k} > \tag{19g}$$

is of the same form as the first term in (19f) and since one takes $\epsilon = 2K/\tau$, by analogy we write

$$\chi_c \langle (\frac{\partial c''}{\partial x_i})^2 \rangle = 2\tau_c^{-1}\Phi \tag{19h}$$

where τ_c is correlation time scale to be discussed below Thus, finally,

$$\frac{D}{Dt}\Phi + D_f(\Phi) = -\Phi_i C_{,i} - 2\tau_c^{-1}\Phi \tag{20}$$

Next, we consider the third term in (2c). Multiply (8a) by u_i, (5e) by c and add the results We obtain.

$$\frac{\partial}{\partial t}(\rho c u_i) + \frac{\partial}{\partial x_j}(\rho c u_i u_j) = F_i c + u_i J_{k,k} \tag{21a}$$

where

$$F_i \equiv -\frac{\partial p}{\partial x_i} - \rho g_i + \frac{\partial}{\partial x_j}\sigma_{ij} \tag{21b}$$

Recalling that

$$\overline{\rho c u_i} = \bar{\rho} U_i C + \overline{\rho u_i'' c''} = \bar{\rho} U_i C + \bar{\rho}\Phi_i \tag{21c}$$

$$\overline{\rho c u_i u_j} = \bar{\rho} U_i U_j C + C\tau_{ij} + U_k \bar{\rho}(\delta_{ik}\Phi_j + \delta_{jk}\Phi_i) + \overline{\rho u_i'' u_j'' c''} \tag{21d}$$

substitution into the mass averaged form of (21a) gives, after several steps,

$$\frac{D}{Dt}\Phi_i + D_f(\Phi_i) = -R_{ij} C_{,j} - \Phi_j U_{i,j} + \bar{\rho}^{-1} A_i \tag{21e}$$

where the function A_i is given by

$$A_i \equiv \overline{F_i c} - F_i C + \overline{u_i'' J_{k,k}} \tag{21f}$$

After some algebra, we have:

$$\overline{F_i c} - F_i C = -g\bar{\rho}\Lambda_i \overline{c''} - \langle c'' \frac{\partial p''}{\partial x_i} \rangle + \overline{F_i(\mathrm{vis}) c} - \overline{F_i(\mathrm{vis})}\, C \tag{22a}$$

where $F_i(\mathrm{vis})$ is the last term in (21b) and the dimensionless function Λ_i is given By:

$$\Lambda_i = H_p P^{-1}\frac{\partial P}{\partial x_i}, \quad H_p = P(g\bar{\rho})^{-1} \tag{22b}$$

After some steps, we have:

$$\overline{F_i(\mathrm{vis}) c} - \overline{F_i(\mathrm{vis})}\, C = \nu\bar{\rho} \langle c'' \frac{\partial^2 u_i''}{\partial x_j^2} \rangle \tag{22c}$$

Since by definition

$$\bar{\rho}\, \overline{c''} = -\overline{\rho' c''} \tag{22d}$$

use of the expansion

$$\frac{\rho'}{\bar{\rho}} = -\alpha_T T'' + \alpha_c c'' \tag{22e}$$

gives

$$\overline{c''} = \alpha\overline{T''c''} - \alpha_c\overline{c''^2} \tag{22f}$$

The $\bar{c}''^2$ term will be approximated with 2Φ given by Eq (19a) while $\overline{T''c''}$ will be computed later. Thus, we have:

$$A_i = -g\bar{\rho}\Lambda_i(\alpha\overline{T''c''} - 2\alpha_c\Phi) - <c''\frac{\partial p''}{\partial x_i}> + \nu\bar{\rho}<c''\frac{\partial^2 u''_i}{\partial x_j^2}> + \overline{u''_i J_{k,k}} \tag{22g}$$

To evaluate the last term in (22g), we use (8c) and obtain

$$\overline{u''_i J_{k,k}} = \chi_c <\rho u''_i \frac{\partial^2 c''}{\partial x_k^2}> \tag{24a}$$

since by definition $\overline{\rho u''_i}=0$ The last two terms in (22g) are therefore

$$\nu\bar{\rho}<c''\frac{\partial^2 u''_i}{\partial x_k^2}> + \chi_c\bar{\rho}<u''_i \frac{\partial^2 c''}{\partial x_k^2}> \tag{24b}$$

which, using mathematical identities, we rewrite as

$$\tfrac{1}{2}\bar{\rho}(\nu+\chi_c)\Phi_{i,kk} - \bar{\rho}(\nu+\chi_c)<\frac{\partial u''_i}{\partial x_k}\frac{\partial c''}{\partial x_k}> + \tfrac{1}{2}\bar{\rho}(\nu-\chi_c)[\frac{\partial}{\partial x_k}(c''\frac{\partial u''_i}{\partial x_k}) - \frac{\partial}{\partial x_k}(u''_i\frac{\partial c''}{\partial x_k})]$$

The first term represents the diffusion of Φ_i. We shall neglect the last term since one can argue that c'' and the velocity gradient peak at different wavenumbers and there is therefore little overlap. As for the second term, it has a structure intermediate between (19g) and (19h) and it will therefore be written as

$$\bar{\rho}(\nu+\chi_c)<\frac{\partial u''_i}{\partial x_k}\frac{\partial c''}{\partial x_k}> = \bar{\rho}\chi_c\tau_{uc}^{-1}\Phi_i \tag{24c}$$

The last term we must compute is the pressure correlation term

$$\Pi_i^c = <c''\frac{\partial p''}{\partial x_i}> \tag{24d}$$

Using the analogy with the temperature case, we write

$$\Pi_i^c = \bar{\rho}\tau_{pc}^{-1}\Phi_i \tag{25e}$$

Finally, the complete equation for Φ_i is

$$\frac{D}{Dt}\Phi_i + D_f(\Phi_i) = -R_{ij}C_{,j} - \Phi_j U_{i,j} - g\Lambda_i(\alpha\overline{T''c''} - 2\alpha_c\Phi) - \tau_{pc}^{-1}\Phi_i + \tfrac{1}{2}\chi_c(1+\nu/\chi_c)\Phi_{i,kk}$$

(26)

where we have absorbed τ_{uc} into τ_{pc}. Next, we consider the fourth function in (2c) which we generalize to

$$\psi = \tfrac{1}{2}\overline{\rho T''^2} = \bar{\rho}\Psi \tag{27a}$$

First, we recall that, except for the last term, the temperature equation (16a) can be treated as in Canuto (1997), where the equation for ψ is given by Eq.(26f). Thus, we must add the last term in (16a),

$$-\frac{25}{8}\chi_c \, Rc_p^{-1}\rho T \frac{\partial c}{\partial x_k}\frac{\partial T}{\partial x_k} \tag{27b}$$

in the derivation, one encounters the term

$$<T''\frac{\partial^2 T''}{\partial x_i^2}> = -<(\frac{\partial T''}{\partial x_i})^2> + \frac{\partial^2}{\partial x_i^2}(\tfrac{1}{2}\overline{T''^2}) \tag{27c}$$

While the last term represents the diffusion of the potential energy $\tfrac{1}{2}T''^2$, the first term represents the dissipation of it and we shall write it in analogy to the dissipation of turbulent kinetic energy, Eq.(19g), with a time scale τ_θ to be discussed later. The final form of the dynamic equation for Ψ is:

$$\frac{D\Psi}{Dt} + \bar{\rho}^{-1}D_f = -c_p^{-1}H_iT_{,i} - 2\tau_\theta^{-1}\Psi + \chi\Psi_{,kk} - \frac{25}{16}\chi_c Rc_p^{-1}\bar{\rho}\frac{\partial C}{\partial x_k}\frac{\partial T^2}{\partial x_k} \tag{27d}$$

where the heat flux H_i is defined in Eq.(16b). Next, we consider the second term in (2c), namely the heat flux (16b) Here too, the relevant dynamic equation was already derived in Canuto (1997), Eq (24a), to which we must add the last term of (16a) In the derivation one encounters a term analogous to (24b), specifically,

$$\nu<T''\frac{\partial^2 u_i''}{\partial x_k^2}> + \chi<u_i''\frac{\partial^2 T''}{\partial x_k^2}> \tag{27e}$$

which we treat in a similar fashion. The final result is:

$$\begin{aligned}\frac{D}{Dt}H_i + c_pD_f(H_i) &= -c_pR_{ij}T_{,j} - H_jU_{i,j} - c_pg\Lambda_i\overline{T''} - \tau_{p\theta}^{-1}H_i \\ &+ \tfrac{1}{2}(\nu+\chi)H_{i,kk} - \frac{25}{8}R\chi_cU_i\frac{\partial C}{\partial x_k}\frac{\partial T}{\partial x_k}\end{aligned} \tag{27f}$$

where the pressure term give rises to the relaxation term $\tau_{p\theta}^{-1}$. Finally, we use the fact that

$$\bar{\rho}\,\overline{T''} = -\overline{\rho' T''} \tag{28a}$$

and the expansion (22e) to obtain

$$\overline{T''} = \alpha \overline{T''^2} - \alpha_c \overline{c'' T''} \tag{28b}$$

so that (27f) becomes:

$$\frac{D}{Dt}H_i + c_p D_f(H_i) = -c_p R_{ij} T_{,j} - H_j U_{i,j} - c_p g \Lambda_i (2\alpha\Psi - \alpha_c \overline{c'' T''})$$
$$- \tau_{p\theta}^{-1} H_i + \tfrac{1}{2}(\nu+\chi) H_{i,kk} - \frac{25}{8} R \chi_c U_i \frac{\partial C}{\partial x_k}\frac{\partial T}{\partial x_k} \tag{28c}$$

Finally, let us consider the last term in (2c), the correlation between T'' and c'' We recall that in general

$$\frac{dc}{dt} = \frac{DC}{Dt} + \frac{Dc''}{Dt} + u_i''\left(\frac{\partial C}{\partial x_i} + \frac{\partial c''}{\partial x_i}\right) \tag{29a}$$

and thus from Eq. (8b)

$$\rho\left(\frac{DC}{Dt} + \frac{Dc''}{Dt} + u_i'' \frac{\partial C}{\partial x_i} + u_i'' \frac{\partial c''}{\partial x_i}\right) = J_{k,k} \tag{29b}$$

Subtracting the mass average of (29b) from (29b) itself, we obtain

$$\frac{Dc''}{Dt} + (u_i'' - \overline{u_i''})\frac{\partial C}{\partial x_i} = \langle u_i'' \frac{\partial c''}{\partial x_i}\rangle - u_i'' \frac{\partial c''}{\partial x_i} + \rho^{-1} J_{k,k} - \langle \rho^{-1} J_{k,k}\rangle \tag{29c}$$

Multiplying (29c) by T'' and mass averaging, we obtain:

$$\langle T'' \frac{Dc''}{Dt}\rangle + (\overline{T'' u_i''} - T''\, \overline{u_i''}) C_{,i} = \langle \rho^{-1} T'' J_{k,k}\rangle - \overline{T''}\langle \rho^{-1} J_{k,k}\rangle + .. \tag{29d}$$

where by ...(higher order terms) we mean all the terms that entail correlations higher than the second–order terms under consideration. For example, if we neglect the h o., we must also neglect $\overline{u_i''}$ in (29d): in fact, because of the second relation in Eqs.(5b), $\overline{u_i''}$ is already a second order variable As for the equation for T'', we employ Eqs (27) and (32) of Canuto (1993; with obvious change in notation) to which we must add the last term in (15a). Keeping only the largest terms, we have

$$\frac{DT''}{Dt} = -u_i'' T_{,i} - (u_i'' T'' - \overline{u_i'' T''})_{,i} + \chi T''_{,kk}$$
$$- \frac{25}{8} R c_p^{-1} \chi_c U_i \left(\frac{\partial c''}{\partial x_k}\frac{\partial T}{\partial x_k} + \frac{\partial C}{\partial x_k}\frac{\partial T''}{\partial x_k}\right) \tag{29e}$$

Once we multiply by c'' and mass average, we obtain:

$$<c''\frac{DT''}{Dt}> = -\overline{c''u_i''}\,T_{,i} + \chi<c''\frac{\partial^2}{\partial x_k^2}T''> + \text{h o} \tag{29f}$$

Adding Eq. (29d) to (29f), we obtain

$$\frac{D}{Dt}\overline{T''c''} = -\Phi_i T_{,i} - c_p^{-1}H_i C_{,i} + \chi<c''\frac{\partial^2}{\partial x_k^2}T''> + <\rho^{-1}T''J_{k,k}> - \overline{T''}<\rho^{-1}J_{k,k}> \tag{29g}$$

The last term becomes

$$\chi_c(\alpha_T\overline{T''^2} - \alpha_c\overline{c''T''})\frac{\partial^2 C}{\partial x_i^2} \tag{29h}$$

whereas

$$\chi<c''\frac{\partial^2}{\partial x_k^2}T''> + <\rho^{-1}T''J_{k,k}> = \chi<c''\frac{\partial^2}{\partial x_k^2}T''> + \chi_c<T''\frac{\partial^2}{\partial x_k^2}c''> \tag{29i}$$

has a form analogous to (24c) and will be treated in similar fashion giving rise to the two most important terms

$$\tfrac{1}{2}\bar{\rho}(\chi+\chi_c)\frac{\partial^2}{\partial x_i^2}\overline{T''c''} - \bar{\rho}(\chi+\chi_c)<\frac{\partial c''}{\partial x_i}\frac{\partial T''}{\partial x_i}> \tag{29j}$$

Thus, finally:

$$\frac{D}{Dt}\overline{T''c''} = -\Phi_i T_{,i} - c_p^{-1}H_i C_{,i} - \tau_{c\theta}^{-1}\overline{T''c''} + \tfrac{1}{2}(\chi+\chi_c)\frac{\partial^2}{\partial x_i^2}\overline{T''c''} - \chi_c(\alpha_T\overline{T''^2} - \alpha_c\overline{c''T''})\frac{\partial^2 C}{\partial x_i^2} \tag{29k}$$

VII. Non–Local Model

To simplify the use of the equations we have derived, we list them below, beginning with the equations for the mean quantities:

Large scale flow, U_i:

$$\bar{\rho}\frac{D}{Dt}U_i = -\frac{\partial}{\partial x_j}(P\delta_{ij} + \bar{\rho}R_{ij}) - \bar{\rho}g_i \tag{30a}$$

Mean temperature T:

$$\bar{\rho}c_p\frac{DT}{Dt} = -(\bar{\rho}H_i + \overline{F_i^r})_{,i} + \omega\frac{DP}{Dt} + \omega(\alpha_T c_p^{-1}H_i - \alpha_c\Phi_i)P_{,i} \tag{30b}$$

where ω is given by (15b)

Mean concentration C:

$$\bar{\rho}\frac{DC}{Dt} = \frac{\partial}{\partial x_i}(\bar{\rho}\chi_c\frac{\partial C}{\partial x_i} - \bar{\rho}\Phi_i) \tag{30c}$$

Reynolds Stresses $\overline{\rho u''_i u''_j} = \bar{\rho}R_{ij}$:

$$\bar{\rho}(\frac{D}{Dt}R_{ij} + D_{ij}) = A_{ij} + B_{ij} - \pi_{ij} - \frac{2}{3}\bar{\rho}\epsilon\delta_{ij} \tag{31a}$$

where

$$-A_{ij} \equiv \bar{\rho}(R_{ik}U_{j,k} + R_{jk}U_{i,k}) \tag{31b}$$

$$B_{ij} = -[c_p^{-1}\alpha_T H_i - \alpha_c\Phi_i]P_{,j} + (i\rightarrow j) \tag{31c}$$

The pressure–velocity correlation π_{ij} is discussed in Appendix A

Turbulent kinetic energy $K=\frac{1}{2}R_{ii}$:

$$\frac{D}{Dt}K+D_f(K) = -R_{ij}U_{i,j} - \bar{\rho}^{-1}[\alpha_T c_p^{-1}H_i - \alpha_c\Phi_i]P_{,i} - \epsilon \tag{32}$$

In both (31a) and (32) we have not included the dilation term $\overline{p'd}$

Convective flux, $c_p\overline{\rho u''_i T''} = \bar{\rho}H_i = c_p\bar{\rho}J_i$:

$$\frac{D}{Dt}H_i + c_pD_f(H_i) = -c_pR_{ij}T_{,j} - H_jU_{i,j} - c_p\bar{\rho}^{-1}[2\alpha_T\Psi - \alpha_c\overline{c''T''}]P_{,i}$$
$$- \tau_{p\theta}^{-1}H_i + \frac{1}{2}(\nu+\chi)H_{i,kk} \tag{33}$$

Temperature fluctuations, $\frac{1}{2}\overline{\rho T''^2} = \bar{\rho}\Psi$:

$$\frac{D\Psi}{Dt} + \bar{\rho}^{-1}D_f(\Psi) = -c_p^{-1}H_iT_{,i} - 2\tau_\theta^{-1}\Psi + \chi\Psi_{,kk} \tag{34}$$

Concentration variance, $\frac{1}{2}\overline{\rho c''^2} = \bar{\rho}\Phi$

$$\frac{D}{Dt}\Phi+D_f(\Phi) = -\Phi_iC_{,i} - 2\tau_c^{-1}\Phi \tag{35}$$

Concentration flux, $\overline{\rho c''u''}_i = \bar{\rho}\Phi_i$:

$$\frac{D}{Dt}\Phi_i+D_f(\Phi_i) = -R_{ij}C_{,j} - \Phi_jU_{i,j} - \bar{\rho}^{-1}(\alpha_T\overline{T''c''}-2\alpha_c\Phi)P_{,i} - \tau_{pc}^{-1}\Phi_i + \frac{1}{2}\chi_c(1+\nu/\chi_c)\Phi_{i,kk} \tag{36}$$

Temperature–concentration correlation, $\overline{T''c''}$:

$$\frac{D}{Dt}\overline{T''c''} + D_f = -\Phi_iT_{,i} - c_p^{-1}H_iC_{,i} - \tau_{c\theta}^{-1}\overline{T''c''} + \frac{1}{2}(\chi+\chi_c)\frac{\partial^2}{\partial x_i^2}\overline{T''c''} -$$
$$- \chi_c(\alpha_T\overline{T''^2}-\alpha_c\overline{c''T''})\frac{\partial^2 C}{\partial x_i^2} \tag{37}$$

The time scales τ_{pc}, $\tau_{c\theta}$, τ_c, $\tau_{p\theta}$, τ_θ will be discussed below.

VIII. Diffusivities. The K–ϵ Model

A widely used turbulence models is the non–local K–ϵ model in which both K and ϵ are treated non–locally while all the remaining turbulence variables are treated locally. The equations for the mean variables are unchanged We have two non–local equations:

Kinetic energy K:

$$\frac{D}{Dt}K + D_f = -R_{ij}U_{i,j} + g\lambda_i(\alpha_T J_i - \alpha_c \Phi_i) - \epsilon \tag{38}$$

Dissipation rate ϵ:

$$\frac{D}{Dt}\epsilon + D_f = -c_s R_{ij}U_{i,j} + c_1 g\lambda_i(\alpha_T J_i - \alpha_c \Phi_i)\epsilon K^{-1} - c_2\epsilon^2 K^{-1} \tag{39}$$

while the other turbulence variables are given by the local expressions

Convective flux, $F_i^C = c_p\overline{\rho u_i'' T''} = c_p\bar{\rho}J_i$:

$$\tau_{p\theta}^{-1} J_i = -R_{ij}T_{,j} - J_k U_{i,k} - \bar{\rho}^{-1}[2\alpha_T\Psi - \alpha_c\overline{c''T''}]P_{,i} \tag{40}$$

Temperature fluctuations, $\frac{1}{2}\overline{\rho T''^2} = \bar{\rho}\Psi$:

$$\Psi = -\tfrac{1}{2}\tau_\theta J_i T_{,i} \tag{41}$$

Concentration variance, $\frac{1}{2}\overline{\rho c''^2} = \bar{\rho}\Phi$.

$$\Phi = -\tfrac{1}{2}\tau_c \Phi_i C_{,i} \tag{42}$$

Concentration flux, $\overline{\rho c'' u''}_i = \bar{\rho}\Phi_i$:

$$\tau_{pc}^{-1}\Phi_i = -R_{ij}C_{,j} - \Phi_j U_{i,j} - \bar{\rho}^{-1}(\alpha_T\overline{T''c''} - 2\alpha_c\Phi)P_{,i} \tag{43}$$

Temperature–concentration correlation, $\overline{T''c''}$:

$$\tau_{c\theta}^{-1}\overline{T''c''} = -\Phi_i T_{,i} - J_i C_{,i} \tag{44}$$

Reynolds stresses (Appendix A):

$$b_{ij} = R_{ij} - \frac{2}{3}K\delta_{ij} \tag{45}$$

$$2\tau_{pv}^{-1}b_{ij} = -\frac{8}{15}KS_{ij} - (1-p_1)\Sigma_{ij} - (1-p_2)Z_{ij} + \beta_5 B_{ij} \tag{46}$$

Solving Eqs.(42)–(44), we obtain:

$$(\delta_{ij} + \eta_{ij})\Phi_j = -d_{ik}\frac{\partial C}{\partial x_k} \tag{47a}$$

where

$$\tau_{pc}^{-1}d_{ik} = R_{ik} + \alpha_T g \tau_{c\theta}\lambda_i J_k \tag{47b}$$

$$\tau_{pc}^{-1}\eta_{ij} = U_{i,j} - g\lambda_i(-\alpha_T\tau_{c\theta}T_{,j} + \tau_c\alpha_c\frac{\partial C}{\partial x_j}) \tag{47c}$$

$$\lambda_i = -(g\bar{\rho})^{-1}\frac{\partial P}{\partial x_i} \tag{47d}$$

Equation (47a) begins to acquire a familiar form but to obtain an explicit form for Φ_i we must apply the Hamilton–Cayley theorem. The result is:

$$\Phi_i = -D_{ij}\frac{\partial C}{\partial x_j} \tag{48a}$$

where the *turbulent diffusivity tensor* D_{ij} is given by:

$$D_{ij} = A(A_0\delta_{ik} + A_1\eta_{ik} + \eta_{im}\eta_{mk})d_{kj} \tag{48b}$$

with

$$A_0 = 1+L_1-L_2,\ A_1 = -1-L_1,\ A = (A_0 + L_3)^{-1} \tag{48c}$$

$$L_1 = \eta_{ii},\quad 2L_2 = -L_1^2 + \eta_{ij}\eta_{ji},$$

$$6L_3 = L_1^3 + 2\eta_{im}\eta_{mk}\eta_{ki} - 3L_1\eta_{ij}\eta_{ji} \tag{48d}$$

From Eqs (42) and (45), we then obtain the expressions for the concentration variance Φ and the $\overline{T''c''}$ correlation:

$$\Phi = \tfrac{1}{2}\tau_c D_{ij}\frac{\partial C}{\partial x_i}\frac{\partial C}{\partial x_j} \tag{49a}$$

$$\overline{T''c''} = -\tau_{c\theta}(J_i - T_{,k}D_{ki})\frac{\partial C}{\partial x_i} \tag{49b}$$

Analogously, using Eqs (41), and (49b) into Eq (40), we obtain an expression for the convective flux J_i which is structurally similar to (47a–c), namely,

$$(\delta_{ik} + \mu_{ik})J_k = -c_{ik}T_{,k} \tag{50a}$$

where.

$$\tau_{p\theta}^{-1}c_{ik} = R_{ik} + \alpha_c g\tau_{c\theta}\lambda_i D_{kj}\frac{\partial C}{\partial x_j} \tag{50b}$$

$$\tau_{p\theta}^{-1}\mu_{ij} = U_{i,j} - g\lambda_i(-\tau_\theta\alpha_T T_{,j} + \tau_{c\theta}\alpha_c\frac{\partial C}{\partial x_j}) \tag{50c}$$

Using the Hamilton–Cayley theorem, we can solve (50a). The convective flux is given by.

$$J_i = -\chi_{ik}T_{,k} \tag{51a}$$

The *turbulent conductivity tensor* χ_{ij} has the following form:

$$\chi_{ij} = B(B_0\delta_{ik} + B_1\mu_{ik} + \mu_{im}\mu_{mk})c_{kj} \tag{51b}$$

with

$$B_0 = 1+M_1-M_2,\ B_1 = -1-M_1,\ B = (B_0 + M_3)^{-1} \tag{51c}$$

$$M_1 = \mu_{ii}, \quad 2M_2 = -M_1^2 + \mu_{ij}\mu_{ji},$$

$$6M_3 = M_1^3 + 2\mu_{im}\mu_{mk}\mu_{ki} - 3M_1\mu_{ij}\mu_{ji} \tag{51d}$$

Finally, Eqs. (48a) for Φ_i and (51a) for J_i must be substituted in Eqs.(A.6–7) so as to obtain the tensor B_{ij}, Eq (A.5) Once that is done, the result is substituted in (46) and the Reynolds stresses R_{ij} can then be obtained in terms of the gradients of the mean variables The solution of (46) entails a system of algebraic equations. We recall that there are only five independent components of R_{ij} since the kinetic energy K satisfies a separate differential equation (32).

IX. No Mean Shear

The 1D case is particularly interesting since it allows a completely analytical solution of the problem. Using Eqs (48a) and (51a) one obtains the heat and concentration (salt) fluxes as

$$\bar{\rho}J_3 \equiv \overline{\rho w''T''} = -\bar{\rho}K_h\frac{\partial T}{\partial z} \tag{52}$$

$$\bar{\rho}\Phi_3 \equiv \overline{\rho w''s''} = -\bar{\rho}K_c\frac{\partial S}{\partial z} \tag{53}$$

The turbulent diffusivities $K_{h,s}$ are given by the expressions:

$$K_h = \nu_T A_h, \qquad K_s = \nu_T A_s \tag{54}$$

where the turbulent viscosity is given by

$$\nu_T \equiv \tau\overline{w^2} \tag{55a}$$

$$A_h = \pi_4(1 + \eta x + \pi_1\pi_2 x R_\rho)D^{-1} \tag{55b}$$

$$A_s = \pi_1(1 + \mu x - \pi_2\pi_4 x)D^{-1} \tag{55c}$$

$$D = (1+\eta x)(1+\mu x) + \pi_1\pi_2^2\pi_4 x^2 R_\rho \tag{55d}$$

$$\eta = \pi_1(\pi_2 - \pi_3 R_\rho), \quad \mu = \pi_4(\pi_5 - \pi_2 R_\rho) \tag{55e}$$

where we have introduced the following dimensionless functions

$$x = \tau^2 N_h^2 \tag{56a}$$

$$\pi_{1,2,3,4,5} = (\tau_{pc}, \tau_{c\theta}, \tau_c, \tau_{p\theta}, \tau_\theta)\tau^{-1} \tag{56b}$$

where N_h^2 has been defined earlier, Eq.(2b). Eqs.(54) are still not the final form since they depend on two unknown variables ν_T and x which we must express in terms of the large scale variables. To compute ν_T, we need an expression for $\overline{w^2}$ For that, we use the equation for the Reynolds stresses, (46), (A.5)--(A.9). We obtain:

$$\nu_T = \frac{1}{3}\epsilon\tau^2[1 + \frac{2}{15}(A_h - A_s R_\rho)x]^{-1} \tag{56c}$$

Next, we need an equation for x. We shall take the local limit of the kinetic energy equation (38) which reads

$$\epsilon = g\alpha_T J_3 - g\alpha_S \Phi_3 = \nu_T N_h^2 (A_s R_\rho - A_h) \tag{57a}$$

Substituting (57a) into (56c), we obtain the equation for x

$$x(A_s R_\rho - A_h) = \frac{15}{7} \tag{57b}$$

which changes (56d) to:

$$\nu_T = \frac{7}{15}\epsilon\tau^2 = \frac{28}{15}\frac{K^2}{\epsilon} \tag{57c}$$

Thus, ν_T is expressed in terms of K and ϵ. Finally, using the expressions for $A_{h,c}$, Eq (57c) becomes

$$A(x)x^2 + B(x)x - \frac{15}{7} = 0 \tag{57d}$$

where A(x) and B(x), which can depend on x (see below), are given by·

$$A = \pi_1(\mu - \pi_2\pi_4)R_\rho - \pi_4(\eta + \pi_1\pi_2 R_\rho) - \frac{15}{7}(\eta\mu + \pi_1\pi_2^2\pi_4 R_\rho) \tag{57e}$$

$$B = \pi_1 R_\rho - \pi_4 - \frac{15}{7}(\eta + \mu) \tag{57f}$$

Thus, x is expressed entirely as a function of R_ρ. Finally, we have

$$K_h = \frac{28}{15}\frac{K^2}{\epsilon}A_h, \quad K_s = \frac{28}{15}\frac{K^2}{\epsilon}A_s \tag{58a}$$

where we still have to determine K and ϵ which in principle are solutions of the two dynamic equation (38) and (39). Eq.(38) has already been used in the local form, that is, Eq (57a) Eq.(39) for ϵ has not yet been used and it can be taken to be local or not Below,

we give the solution corresponding to the case where (39) is taken to be local which means

$$\epsilon = \Lambda^{-1}K^{3/2} \tag{58b}$$

where Λ is a mixing length; the specification of Λ is the price that one has to pay for not solving Eq.(39). From the definition of x, Eq.(56a), and the definition $\tau=2K\epsilon^{-1}$, we obtain, using (58b),

$$K = 4\Lambda^2 N_h^2 x^{-1} \tag{58c}$$

and thus the final expressions for the diffusivities $K_{h,c}$ follow from Eqs.(58a):

$$K_h = \frac{56}{15}\Lambda^2\left(\frac{N_h^2}{x}\right)^{\frac{1}{2}}A_h, \qquad K_s = \frac{56}{15}\Lambda^2\left(\frac{N_h^2}{x}\right)^{\frac{1}{2}}A_s \tag{58d}$$

Thus, the problem is completely solved analytically. In fact, both diffusivities are now expressed in terms of the gradients $\partial T/\partial z$ and $\partial S/\partial z$ Clearly, when $N_h^2<0$ (corresponding to unstable stratification), x must be taken as the negative solution of (57e) since K is always positive, Eq (58c)

In addition to the heat and salt turbulent diffusivities, it is also useful to introduce a mass diffusivity K_ρ. We begin by using Eqs.(2b) and (22e) to rewrite (38) as

$$\frac{DK}{Dt} + D_f(K) = K_m N_u^2 - g\bar{\rho}^{-1}\overline{\rho' w'} - \epsilon \tag{59a}$$

where

$$F_\rho \equiv \overline{\rho' w'} \tag{59b}$$

is the "mass flux". Quantifying the strength of shear by the dimensionless parameter Γ (Hamilton et al., 1989)

$$\Gamma = K_m N_u^2 \epsilon^{-1} - 1 \tag{59c}$$

we have in the stationary and local case

$$\overline{\rho' w'} = \bar{\rho} g^{-1} \Gamma \epsilon \tag{59d}$$

If we further write the mass flux as

$$\overline{\rho' w'} = -K_\rho \frac{\partial \rho}{\partial z} \tag{59e}$$

the "*turbulent mass diffusivity*" K_ρ becomes (Schmitt, 1994, Eq.7):

$$K_\rho = \Gamma\frac{\epsilon}{N^2}, \qquad N^2 = N_h^2(1-R_\rho) \tag{59f}$$

On the other hand, use of Eqs.(22e), (52) and (53) gives

$$\frac{g}{\rho}\overline{\rho' w''} = N_h^2(K_h - K_c R_\rho) \tag{59g}$$

Using Eqs.(59e) and (2b), we obtain the following expression for K_ρ in terms of K_h and K_c:

$$K_\rho = (K_h - K_c R_\rho)(1-R_\rho)^{-1} \tag{60}$$

In the absence of mean shear, $\Gamma=-1$, Eq (59d) shows that the mass flux is downward

$$\overline{\rho' w''}<0 \tag{61}$$

Thus we have, using (2d–g):

SF, Stable, $R_\rho<1$:

Eqs.(59f) and (60) imply that:

$$K_\rho<0, \quad \frac{K_h}{K_c} < R_\rho < 1 \tag{62a}$$

SF, Unstable $R_\rho>1$

Eqs (59f) and (60) imply that:

$$K_\rho>0, \qquad R_\rho > \frac{K_h}{K_c}, \qquad R_\rho>1 \tag{62b}$$

DC, Stable $R_\rho>1$:

Eqs (59f) and (60) imply that:

$$K_\rho<0, \qquad \frac{K_h}{K_c} > R_\rho>1 \tag{62c}$$

The requirement of dynamical stability $N^2>0$ sets the lower limit for R_ρ while the requirement of turbulent mixing sets the upper limit of R_ρ. This is a natural result since transgressing the upper limit would mean that $\partial S/\partial z$, which acts like sink, is too strong for turbulent mixing to survive.

DC, Unstable, $R_\rho<1$:

Eqs.(59f) and (60) imply that:

$$K_\rho>0, \quad \frac{K_h}{K_c} > R_\rho, \quad R_\rho<1 \tag{62d}$$

X. Qualitative Results. No Shear

Before presenting the numerical solutions of the model, we present some qualitative results. Using the definitions of $K_{h,s}$, Eqs.(58d) and (57c), we derive the relations.

$$\frac{K_h}{K_s} = R_\rho - \frac{15}{7}\frac{1}{xA_s} \tag{63a}$$

In *diffusive–convection*, $x<0$ and since in the stable case $R_\rho>1$, we conclude that

$$K_h>K_s \tag{63b}$$

in accordance with the measurements (Kelley, 1984). In *salt fingers*, we write (57c) as.

$$\frac{K_s}{K_h} = R_\rho^{-1}(1+\frac{15}{7}\frac{1}{xA_h}) \tag{64a}$$

Since $x>0$ and $R_\rho<1$, it follows that

$$K_s> K_h \tag{64b}$$

in agreement with the measurements (Hamilton et al , 1989, fig.2). Furthermore, in *diffusive–convection*, the flux ratio

$$R_F = \frac{\alpha_c \Phi_3}{\alpha_T J_3} = \frac{K_s}{K_h} R_\rho = R_\rho(R_\rho - \frac{15}{7} x^{-1}A_s^{-1})^{-1} \tag{65a}$$

is predicted to be $(x<0)$

$$R_F<1 \tag{65b}$$

in agreement with the data (Kelley 1990, fig.2). Similarly, in *salt fingers* we derive that the flux ratio:

$$R_F = \frac{\alpha_T J_3}{\alpha_c \Phi_3} = \frac{K_h}{K_s} R_\rho^{-1} = (1+\frac{15}{7}x^{-1}A_h^{-1})^{-1} \tag{66a}$$

is predicted to be $(x>0)$

$$R_F<1 \tag{66b}$$

in accord with the measurements (Turner, 1967, fig.4; Schmitt, 1979, fig.4; McDougall and Taylor, 1984, fig.4; Taylor and Buchens, 1989, fig.6; Ozgokmen et al. 1998, fig.13).

XI. Salt–Fingers and Diffusive–Convection. The effect of Shear .

Here, we present the analytic solutions for the turbulent diffusivities of momentum, heat and salt in the presence of the three gradients ∇U ∇T, and ∇C which we take in the

form given of Eqs.(73a,b). K and ϵ can be treated either locally or not. It is convenient to introduce the following dimensionless variables:

$$n_i \equiv -\pi_2\pi_3 g\alpha_T\tau^2\frac{\partial T}{\partial x_i} \tag{67a}$$

$$c_i \equiv \pi_3^2 g\tau^2\alpha_s\frac{\partial S}{\partial x_i} \tag{67b}$$

$$\psi_i \equiv \beta_5 K^{-1}\alpha_T g\tau_{pv} J_i \tag{67c}$$

$$\phi_i \equiv \beta_5 K^{-1}\alpha_c g\tau_{pv}\,\Phi_i \tag{67d}$$

$$\lambda_i = -(g\bar{\rho})^{-1}P_{,i} \tag{67e}$$

eqs.(46), (47a–d) and (50a–c) become:

Reynolds stresses:

$$a_{ij} \equiv K^{-1}R_{ij}-\frac{2}{3}\delta_{ij} \tag{68a}$$

$$2a_{ij} = -\frac{8}{15}\tilde{\Sigma}_{ij} - (1-p_1)\tilde{\Omega}_{ij} - (1-p_2)\tilde{Z}_{ij} + \Psi_{ij} - T_{ij} \tag{68b}$$

$$\Psi_{ij} \equiv \lambda_i\psi_j + \lambda_j\psi_i - \frac{2}{3}\lambda_k\delta_{ij}\psi_k \tag{68c}$$

$$T_{ij} \equiv \lambda_i\phi_j + \lambda_j\phi_i - \frac{2}{3}\lambda_k\delta_{ij}\phi_k \tag{68d}$$

Concentration Flux:

$$(\delta_{ik}+\eta_{ik})\phi_k = -[p_4(a_{ik}+\frac{2}{3}\delta_{ik}) + p_5\lambda_i\psi_k)]c_k \tag{69a}$$

$$\eta_{ij} = p_3\tilde{U}_{i,j} - \lambda_i p_{11}(n_j + c_j) \tag{69b}$$

Temperature Flux:

$$(\delta_{ik}+\mu_{ik})\psi_k = p_6(a_{ik}+\frac{2}{3}\delta_{ik} - p_7\lambda_i\phi_k)n_k \tag{70a}$$

$$\mu_{ij} = p_8\tilde{U}_{i,j} - \lambda_i(p_9 n_j + p_{10}c_j) \tag{70b}$$

where,

$$\begin{gathered}\tilde{U}_{i,j} \equiv \tau_{pv}U_{i,j}, \qquad \tilde{\Sigma}_{i,j} \equiv \tau_{pv}\Sigma_{ij} \\ \tilde{\Omega}_{ij} = \tau_{pv}\Omega_{ij}, \qquad \tilde{Z}_{ij} = \tau_{pv}Z_{ij}\end{gathered} \tag{71}$$

The functions p's are defined as follows:

$$\begin{gathered}p_1=0.832, \quad p_2=0.545, \quad p_3=\frac{5}{2}\pi_1, \quad p_4=\frac{1}{5}\pi_1\pi_3^{-2} \\ p_5=\pi_1\pi_2\pi_3^{-2}, \quad p_6=\frac{1}{5}\pi_3^{-1}\pi_2^{-1}\pi_4, \quad p_7=5\pi_2 \\ p_8=\frac{5}{2}\pi_4, \quad p_9=\pi_5\pi_4(\pi_3\pi_2)^{-1}\end{gathered}$$

$$p_{10} = \pi_2\pi_4\pi_3^{-2}, \quad p_{11} = \pi_1\pi_3^{-1} \tag{72}$$

If we take.

$$\frac{\partial}{\partial x_i}(T,C) \to \delta_{13}\frac{\partial}{\partial z}(T,C), \quad \mathbf{U} = [U(z), V(z), 0] \tag{73a}$$

and thus·

$$\Sigma_{ij} = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & \partial \mathrm{U}/\partial z \\ 0 & 0 & \partial \mathrm{V}/\partial z \\ \partial \mathrm{U}/\partial z & \partial \mathrm{V}/\partial z & 0 \end{bmatrix}, \quad \mathrm{V}_{ij} = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & \partial \mathrm{U}/\partial z \\ 0 & 0 & \partial \mathrm{V}/\partial z \\ -\partial \mathrm{U}/\partial z & -\partial \mathrm{V}/\partial z & 0 \end{bmatrix} \tag{73b}$$

we can give a complete algebraic solution of the algebraic set of equations (68)–(70). Since we are dealing with only one component of the vectors n_i, c_i, we simplify the notation to.

$$n_3 \equiv n = -n_0 g\alpha_T\tau^2\frac{\partial T}{\partial z}, \quad n_0 = \pi_2\pi_3 \tag{74a}$$

$$c_3 \equiv c = c_0 g\tau^2\alpha_c\frac{\partial C}{\partial z}, \quad c_0 = \pi_3^2 \tag{74b}$$

The dimensionless shear is given by:

$$\psi = (\tau_{pv}N_u)^2, \qquad N_u^2 = \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \tag{74c}$$

If we introduce the simplifying notation

$$w \equiv \bar{\rho}^{-1}\rho u_z, \ \theta \equiv T'' \tag{74d}$$

we obtain the following results:

Momentum Flux.

$$\overline{uw} = -K_m\frac{\partial U}{\partial z}, \quad K_m = 2\frac{K^2}{\epsilon}S_m \tag{75a}$$

Heat flux.

$$\overline{w\theta} = -K_h\frac{\partial T}{\partial z}, \quad K_h = 2\frac{K^2}{\epsilon}S_h \tag{75b}$$

Salt Flux:

$$\overline{ws''} = -K_s\frac{\partial S}{\partial z}, \quad K_s = 2\frac{K^2}{\epsilon}S_s \tag{75c}$$

The dimensionless structure functions $S_{m,h,s}$, see Eqs.(3d), are given by

$$S_m = \frac{8}{75}A_m D^{-1}, \ S_h = \frac{4}{15}\pi_4 A_h D^{-1}, \ S_s = \frac{4}{15}\pi_1 A_s D^{-1} \tag{76a}$$

$$A_m = 12 + a_1 n^2 + a_2 nc + a_3 c^2 + a_4 n + a_5 c \tag{76b}$$

$$A_h = (1 + b_1 c + b_2 n)(60 + b_3 y + b_4 c + b_5 n) \tag{76c}$$

$$A_c = (1 + b_6 c + b_7 n)(60 + b_3 y + b_4 c + b_5 n) \tag{76d}$$

$$D = 24 + d_1yn^2 + d_2ync + d_3yc^2 + d_4n^3 + d_5n^2c + d_6nc^2 + d_7c^3 +$$
$$+ d_8yn + d_9yc + d_{10}n^2 + d_{11}nc + d_{12}c^2 + d_{13}y + d_{14}n + d_{15}c \tag{76e}$$

As one can see, the dimensionless functions A's and D depend on the gradients of the mean temperature, concentration and mean *velocity* represented by n, c and y. The functions a_k, b_k and d_k (Appendix C) depend on the time scales τ_c, τ_{pc}, etc which in turn depend on the Peclet numbers. For large Peclet numbers, a_κ, b_κ and d_κ become constant, Appendix C. As before, the variables K and ϵ are in principle solutions of Eqs.(38) and (39). The (superficial) algebraic complexity of the functions A's is a small price to pay when one considers that the above equations are the solution of a fully turbulent problem in the presence of three external fields, T, U and C. It is indeed quite surprising that such a complex problem could be expressed via a set of algebraic relations.

In the case of a local model, Eq.(38) becomes:

$$-R_{ij}U_{i,j} + g\alpha_T\lambda_iJ_i - g\alpha_s\Phi_i = \epsilon \tag{77a}$$

Using the definition of the K's and that of ψ given in Eq.(74c), we have

$$\psi[S_m - Ri(1-R_\rho)^{-1}(S_h - R_\rho S_s)] = \frac{8}{25} \tag{77b}$$

Once we substitute the functions $S_{m,h,s}$, Eq (77c) yields the function:

$$\psi = \psi(Ri, R_\rho) \tag{78a}$$

We recall that in the functions $A_{m,h,s}$ we must substitute:

$$n = -\frac{25}{4}n_0\psi Ri(1-R_\rho)^{-1} \tag{78b}$$

$$c = \frac{25}{4}c_0\psi RiR_\rho(1-R_\rho)^{-1} \tag{78c}$$

We shall exhibit the turbulent diffusivities K_m, K_h and K_s in units of Λ^2N_u (for different values of R_ρ) vs.Ri which we recall is defined as follows:

$$Ri = (g\alpha_T\frac{\partial T}{\partial z})Nu^{-2} \tag{78d}$$

which helps us differentiate between stable and unstable stratification.

XII. The RNG method to determine the time scales τ_{pc}, $\tau_{c\theta}$, τ_c, $\tau_{p\theta}$, τ_θ

To make the above equations predictive, one must know the dissipation time scales of

the different turbulent variables, namely τ_{pc}, $\tau_{c\theta}$, τ_c, $\tau_{p\theta}$, τ_θ. Not surprisingly, this is one of the most difficult problems since one–point closure models, like the one we have used, are unable to provide them. In most engineering and geophysical applications (e.g., the MY model), it was always assumed that

$$\pi_\kappa = (\tau_{pc}, \tau_{c\theta}, \tau_c, \tau_{p\theta}, \tau_\theta)\tau^{-1} \sim \text{constant} \tag{79a}$$

However, on physical grounds, it is only possible to say that

$$\tau_{p\theta} = \tau_{pc}, \quad \tau_\theta = \tau_c \tag{79b}$$

while $\tau_{c\theta}$ remains to be determined. Since in principle, one may want to consider regimes in which the Peclet number of both the temperature and salinity fields are not excessively lager than unity (Pe~1 correspond to low levels of turbulence), we adopt expressions for that were previously depermined: theye are

$$\begin{gathered}(\tau_{p\theta}, \tau_{pc})\tau^{-1} = a\mathrm{Pe}(1+b\mathrm{Pe})^{-1} \\ 4\pi^2 a = 1, \quad 5a(1+\sigma_t^{-1})^{-1}\end{gathered} \tag{79c}$$

$$\begin{gathered}(\tau_\theta, \tau_c)\tau^{-1} = a\mathrm{Pe}(1+a\mathrm{Pe}\sigma_t^{-1})^{-1} \\ 7\pi^2 a = 4\end{gathered} \tag{79d}$$

$$\begin{gathered}\tau_{c\theta}\tau^{-1} = a\mathrm{Pe}_\theta(1+b\sigma_{t\theta}^{-1}\mathrm{Pe}_\theta)^{-1} \\ 7\pi^2 a \equiv 4(1+\mathrm{Pe}_\theta/\mathrm{Pe}_c)^{-1} \\ 4b = 15a(1+\sigma_{t\theta}/\sigma_{tc})\end{gathered} \tag{79e}$$

We have used only one symbol for both Pe and σ_t but clearly in each specific case one must insert the corresponding Pe and σ_t, where:

$$\mathrm{Pe}_{\theta,c} = \frac{4\pi^2}{125}\frac{K^2}{\epsilon}(\chi_\theta^{-1}, \chi_c^{-1}) \tag{79f}$$

where $\chi_{\theta,c}$ are the molecular diffusivities of the two fields. The turbulent Prandtl numbers σ_t are functions of the corresponding Pe and the RNG method gives the following result (Canuto and Dubovikov, 1996)

$$\gamma_2\sigma_t^{-1} = 1 + \frac{2}{5}\pi^2\gamma_2\mathrm{Pe}^{-1}\{[1 + \frac{5}{2\pi^2}\mathrm{Pe}(\sigma_t^{-1}+\gamma_1^{-1})^{-\Gamma} - 1\} \tag{79g}$$

The constants $\gamma_{1,2}$ are given by:

$$2\gamma_1 = (\gamma^2+4\gamma)^{\frac{1}{2}}-\gamma, \quad \gamma_2=\gamma_1+\gamma, \; \Gamma=\gamma_1/\gamma_2, \; \gamma=0.3 \qquad (79h)$$

XIII. Numerical Results

In Figs.1–3 we plot $K_{m,h,s}$ vs. Ri (defined in Eq.78d) for different R_ρ (defined in Eq.2a). The panels are characterized by the symbols SF (salt fingers), DC (diffusive convection), DS (doubly stable) and DU (doubly unstable) defined in Eqs (2d–g). Consider the case of salt–fingers in Fig.1a. At a fixed Ri, the diffusivity increases as R_ρ increases which is physically understandable since the instability is generated by salt and thus the larger the source, the larger the diffusivity. Next, consider the dependence on Ri. We notice that the smaller the shear $Ri \to \infty$, the larger are the K's, which at first may seem paradoxical: since both salt and shear contribute to the instability, their effect should add up What we find is that the larger the shear, the smaller the diffusivity, which implies that shear and salt–fingers work in opposite directions. It is in fact known (Linden 1971, 1974b, Kunze, 1990) that shear has the tendency to disrupt the fingers transport process In the case of DS and DU, Fig.1b, R_ρ is negative, see Eqs.(2f–g). Quite understandably, the former case (right panel) corresponds to the lowest diffusivity because of the large stability introduced by both salt and temperature. The only source of instability is shear and thus turbulent mixing dies when stratification is too strong. In the DU sade, the opposite occurs in the sense that both T and S are unstable and the resulting diffusivities are the largest. The same considerations hold true for K_h and K_s which are shown in Figs.2,3. Consider now the DC case, Fig.1a. At a given Ri, the diffusivity decreases as R_ρ increases, the opposite of the SF case. This is in accordance with the fact that in this case salt acts as a sink of turbulent mixing (which is caused by an unstable temperature gradient), and thus, the stronger the sink, the lower the level of turbulence, a circumstance that is reflected in the decrease of the diffusivity. As for the effect of shear, we notice that here too, the smaller the shear (large Ri), the larger the diffusivities which implies that shear prevents the mixing caused by the temperature instability. However, this is not true in general. the

curves first decrease with increasing Ri, which indicates that for moderate Ri shear helps mixing, as one would expect, but the trend does not continue since the curves change curvature. However, there is saturation phenomenon which does not occur in the SF case At large R_ρ (large sink), the help in mixing from shear saturates.Finally, the lowest three curves correspond to a stable situation, while the second and third correspond to an unstable situation. In Figs. 2b and 3b we present K_h and K_s in the DU (doubly unstable) and DS (doubly stable) cases. Quite naturally, in the altter case the diffusivities are the lowest. In Figs.4–6 we plot the ratios K_m/K_h, K_m/K_s and K_h/K_s which show quite clearly that the diffusivities are indeed different among themselves. In Figs.7–10 we exhibit the turbulent mass diffusivity K_ρ defined in Eq (59e) and given in terms of $K_{h,s}$ by Eq (60). In Fig 11 we plot Γ defined in Eq.(59f). Schmitt (1994) "measured" values of Γ=0 18–0.25 are indeed predicted by the model for the case of salt fingers (upper right panel) for quite a range of Ri but the precise value depends on R_ρ.

The length scale Λ is determined using the Deardorff–Blackadar formula:

$$\Lambda = 2^{-3/2} B_1 \ell, \qquad B_1 = 24.7,$$

$$\ell = \min(\tfrac{1}{2}\tfrac{q}{N}, \ell_1) \tag{80a}$$

$$\ell_1 = \kappa z \ell_0 (\ell_0 + \kappa z)^{-1}, \qquad \ell_0 = 0.17\mathrm{H} \tag{80b}$$

where $\frac{1}{2}q^2$=K is the turbulent kinetic energy, N is the Brunt–Vaisala frequency, κ=0 4 is the von Karman constant and H is the mixed layer depth. When used within the NCAR CSM Ocean Model, H is determined as the depth where the buoyancy difference

$$g[\rho(H) - \rho(\text{surface})]\rho(H)^{-1} = 3\ 10^{-4}\mathrm{ms}^{-2} \tag{80c}$$

XIV. Ocean GCM

We tested the new vertical diffusivities in a global ocean general circulation model, the NCAR CSM Ocean Model produced by the University Corporation for Research, National Center for Atmospheric Research, Climate and Global Dynamics Division. They developed their model by modifying the MOM 1.1 GFDL code (NCAR CSM Ocean Model

Technical Note, The NCAR CSM Ocean Model, by the NCAR Oceanography section). We employed the stand–alone 3^0x3^0 configuration of the model detailed in their technical note with the default parameter values. It has 3.6^0 spacing in longitude and a variable spacing in latitude increasing from 1.8^0 at the equator to 3.4^0 at 17^0 N, S and then decreasing back to 1.8^0 for 60^0 N, S and poleward. There are 25 levels of increasing thickness in the vertical, with the surface level 6 meters thick The option for the GM mesoscale eddy parameterization was enabled. Bulk forcing with a seasonal cycle plus a 1/2 year timescale restoring condition on the salinity is used, except under sea–ice where there is strong restoring. This configuration corresponds to the case B–K described in Large et al (1997). It should be noted, however, that for determination of the length scale in our turbulence model we used the program's definition for mixed layer depth (a buoyancy difference from the surface of 3 10^{-4}ms^{-2}), which is different from that graphed as a diagnostic in Fig.5 of Large et al.(1997) We initialized our runs with annually averaged Levitus data and ran for 126 momentum years. As in Large et al. (1997) a 3504sec timestep for momentum is used, while for the first 96 momentum years the tracers are accelerated by a factor increasing from 10 at the surface to 100 for the deep ocean. We then set all timesteps equal for the remaining 30 years as they did.

First, we ran the NCAR program as is, with the option for the KPP mixing enabled, producing the KPP data presented in the figures below. Then, in place of the KPP module, we inserted a module which uses our new model for the diffusivities for momentum and heat with the salt diffusivity set equal to that of heat. To save computing time, we constructed tables of the dimensionless functions $S_{m,h}$ and of the dimensionless variable y (obtained from solving Eq.68e),

$$y = \tfrac{1}{2}\frac{S^2\ell^2}{K} \equiv \tfrac{1}{2}x^2\left(\frac{\ell}{\Lambda}\right)^2 \qquad (81a)$$

vs. Ri. Then, for each point in space and time these were interpolated to the local Ri. The diffusivities $K_{m,h}$ were written in terms of (81a) as

$$K_{m,h}/\ell^2 S = \tfrac{1}{2}B_1 y^{-\frac{1}{2}} S_{m,h} \tag{81b}$$

XV. Below the Mixed Layer

Below the ocean mixed layer, the external wind–generated shear is too small to generate turbulent mixing and yet, even in regions where both the temperature and the salinity gradients are stably stratified, it is usual to assume "background diffusivities" for viscosity, heat and salt diffusivity which are believed to be caused by internal wave breaking (Large et al., 1997). In our case, when we assumed $K_h=K_s$, we followed the same practice. It would be preferable not to do so but rather model the physical processes causing this background mixing. Our main assumption is that the turbulence model has given us the correct functional dependence of the $K_{m,h,s}$ on Ri and R_ρ and that such diffusivities can thus be used below the ML. Since all the arguments discussed below, are valid for any of the three K's, we shall use only the generic symbol K and write succinctly

$$K = K(Ri, R_\rho) \tag{82a}$$

The key problem is how to define and thus compute Ri. Here, we shall make us of the measured data (Gargett et al., 1981) on the vertical shear generated by the wave breaking phenomenon. By integrating over all wavenumbers one can compute the shear due to internal waves, S_{wb}. One can then form a corresponding Ri_{wb} as follows:

$$Ri_{wb} = N^2/S^2_{wb} \tag{82b}$$

where

$$N^2 = -g\rho^{-1}\frac{\partial\rho}{\partial z} \tag{82c}$$

Gargett et. al. (1981, sec. 5) confirmed earlier arguments by Munk (1966) that Ri_{wb}~1 To those argument, we would like to add the following consideration. As the value of Ri_{cr}, above which there is no longer turbulent mixing, computed from our model is O(1), if Ri_{wb} were >>1, there would be no turbulence generated by the internal waves at all. On the other hand if Ri_{wb} were <<1, there would be a very strong turbulence producing a viscosity sufficient to damp out the waves themselves. The wave–generated turbulence is thus self–limiting Since the turbulence model gives a precise value for Ri_{cr}, while the

above argument only tells us that $Ri_{wb} \sim O(1)$, we shall write:

$$Ri_{wb} = cRi_{cr} \tag{82d}$$

where c is a constant reasonably close to unity. We have found that c=0.88 gives a diffusivity close that measured by Ledwell et al. (1993). Since in the local model, the K's are also proportional to the length scale Λ or ℓ. Below the mixed layer, we thus need an analogous ℓ_{wb}. We shall use the same formal expressions (80a,b) but with different $\ell_0(wb)$ which we compute as follows. Assuming a Kolmogorov spectrum at wavenumbers upward of a breakpoint k_0 and integrating, we obtain:

$$\ell_0(wb) = (3Ko)^{3/2}(B_1 k_0)^{-1} \tag{82e}$$

where Ko=1.6 is the Kolmogorov constant. We identify k_0 with the best value of Gargett et al. (1981) for the break in slope of the observed spectrum of internal waves, namely

$$k_0 = \frac{1}{10}\, 2\pi \text{ radians/meter} \tag{82f}$$

Thus, $\ell_0(wb)$ is known and so is ℓ_{wb}. Similarly, y_{wb} is obtained by solving the production=dissipation, Eq (77b). Thus, the complete wave–breaking expressions for the three diffusivities are:

$$K_{m,h,s}(wb) = \tfrac{1}{2}B_1\ell^2(wb)S_{wb}y_{wb}^{-\frac{1}{2}}\, S_{m,h,s}(Ri_{wb}, R_\rho) \tag{82g}$$

We add together the diffusivities calculated using the shear resolved in the ocean model and the background diffusivities, ensuring continuity in the transition between regions where external excited shear dominates and those where the internal wave shear does We thus take the total diffusivities to be:

$$K_{m,h,s} = K_{m,h,s}(Ri,R_\rho) + K_{m,h,s}(wb, R_\rho) \tag{82h}$$

In the statically unstable case (Ri< 0), we set $K_{m,h,s}(wb)=0$. The very large mixing due to convective instability makes the background irrelevant in this situation in any case.

XVI. O–GCM results

Using the model for the K's extended all the way to the bottom of the ocean, we obtain the results presented in Figs.12–23. In each case, we compare the results with

Levitus (1994) data, with the KPP model ($K_h=K_s$) for which we have rerun the code and with our model with $K_s=K_h$. In Figs. 24–32 we plot $K_{m,h,s,\rho}$ (cm^2s^{-1}) vs. depth (meters) at different locations. As expected, the K's are small below the mixed layer where they can reach very high values, as we explicitly show in Figs.30–32. In case of the Canary Islands, Fig.29, the diffusivity of a truly passive scalar (and thus strictly not $K_{m,h,s,\rho}$) was measured by Ledwell et al. (1993) to have a value of 0.11 ± 0.02 cm^2s^{-1}. Finally, in Fig.33 we present the polar heat transport. As already discussed in the work of MHG, global properties are not strongly affected by double diffusion phenomena.

XVII. Conclusions.

Considering the importance of double diffusion phenomena in oceanography (Schmitt, 1994, Zhang et al., 1998; Merrifield et al., 1999), we believe we have made a quantitative step by presenting a new formalism. The resiliency of the new approach is demonstrated by the fact that it can encompass salt–fingers, diffusive–convection, doubly stable and doubly unstable gradients The whole formalism was developed so as to include shear which, though of different origin at differernt depths, is always present. Within the mixed layer, it is mainly due to external wind gradients while in the ocean interior is believed to be mainly due to wave breaking phenomena.

Clearly, the model with salinity would have lost much of its attractiveness if we could use it only in the ML and if we had to parameterize the physical processes below the ML with adjustable background diffusivities as done thus far. We have suggested that below the ML the functional dependence of the three diffusivities on the two stability parameters R_ρ and Ri is still the one given by the turbulence model since the latter does not depend on any specific form of the shear entering the Richardson number Ri. As dicvussed in XIV, we have used the data on vertical shear measured by Gargett et al. (1981).

The final model comes in more than one flavor depending on whether on uses local or non–local models. Logically, the first model we have treid is the local one since the whole

problem can be solved analytically. The expressions for the turbulent diffusivities are algebraic. The whole turbulence problem is reduced to the solution of a cubic equation

The problem is however far from solved. Of particular significance is the role played by the salinity–temperature correlation. If we were to assume that

$$\overline{T''S''} = (\overline{T''^2})^{\frac{1}{2}}(\overline{S''^2})^{\frac{1}{2}} \tag{83a}$$

and that

$$\tau_{c\theta} = \tfrac{1}{2}\tau_c = \tfrac{1}{2}\tau_\theta \tag{83b}$$

as one may be tempted to do, one would obtain that the two fields are indistinguishable and this implies that

$$K_s = K_h \tag{83c}$$

contrary to what is observed. Fortunately, the present model dos not require either of Eqs (83a,b) but once they are imposed, (83c) follows. These and similar questions will be the subject of future studies.

Figure caption

Fig.1a Momentum diffusivity K_m in units of $\Lambda^2 Nu$, see Eqs.(74c) and (58b) vs. Ri defined in Eq.(78d) for different values of the Turner number R_ρ defined in Eq.(2a). The label DC and SF are defined in Eqs.(2d)–(2e).

Fig.1b Same as in Fig.1a for the DU and DS cases, Eqs.(2d)–(2e).

Fig.2a. Heat diffusivity vs. Ri for different R_ρ. Salt–fingers and diffusive–convection

Fig.2b Same as in Fig.2a for the DU and DS cases.

Fig.3a Salt diffusivity K_s vs. Ri for the SF and DC cases

Fig.3b Same as in Fig.3b for the DU and DS cases

Fig 4 The turbulent Prandtl number K_m/K_h vs. Ri. The heavy line corresponds to the laboratory data discussed in paper II, Figs.3,4.

Fig 5. The ratio of momentum to salt diffusivity vs. Ri for different R_ρ

Fig.6 The ratio of heat to salt diffusivity vs. Ri for the DC and SF cases for different R_ρ

Fig 7 The mass flux diffusivity K_ρ defined in Eqs.(59e) and (60) vs. Ri for different R_ρ for the DC and SF cases

Fig.8 The ratio K_ρ/K_m vs. Ri for different R_ρ for the DC and SF cases.

Fig.9 Same as in Fig.8 for the ratio K_ρ/K_h

Fig 10 Same as in Fig.8 for the ratio K_ρ/K_s

Fig 11 The efficiency parameter Γ defined in Eqs.(59c) and (59f) vs. Ri for different values of R_ρ. A value Γ=0.18–0.25 (Schmitt, 1994) is indeed predicted by the model for the salt–finger case.

Fig.12 The resulting global ocean temperature using the O–GCM discussed in XIV with the background diffusivities computed following the new procedure discussed in XIV above. The Levitus (1994) data are the solid line. We have also run the O–GCM code with the KPP model (K_s=K_h) and the results are indicated by diamonds. The results with our new model with K_s=K_h are shown by squares while the full model with $K_s \neq K_h$ are indicated by

asterisks.

Fig.13 Same as Fig.12 for the global salinity

Fig.14. Same as Fig.12 for the Artic ocean

Fig.15. Same as Fig 13 for the Artic ocean

Fig.16 Same as Fig.12 for the Atlantic ocean

Fig.17. Same as Fig.13 for the Atlantic ocean

Fig.18. Same as Fig.12 for the Pacific ocean

Fig.19. Same as Fig.13 for Pacific ocean

Fig.20. Same as Fig.12 for the Indian ocean

Fig.21. Same as Fig.13 for the Indian ocean

Fig.22. Same as Fig.12 for the Southern ocean

Fig 23 Same as Fig.13 for the Southern ocean

Fig.24. The foir diffusivities $K_{m,h,s,\rho}$ (cm^2s^{-1}) for the Papa staion

Fig.25 Same as in Fig.24 for the Artic ocean

Fig.26 Same as in Fig.24 for the Canary Islands.

Fig 27 Same as in Fig 24 but for the first 1km

Fig 28 Same as in Fig.25 for the first 1km

Fig.29 Same as in Fig 26 for the first 1km. Ledwell et al. (1993) value of 0.11 ± 0.02 cm^2s^{-1} (see, however, the discussion in the main text)

Fig.30. Same as in Fig.27 for the first 40m

Fig.31 Same as in Fig.28 for the first 40m

Fig.32 Same as in Fig.29 for the first 60m.

Fig.33 Polar heat transport vs. latitude for three different models.

Appendix A: Reynolds stress equations

Rather than employing Eq.(31a), we introduce the traceless tensor

$$b_{ij} = R_{ij} - \frac{1}{3}\delta_{ij}R_{kk} = R_{ij} - \frac{2}{3}\delta_{ij}K \tag{A 1}$$

where K satisfies Eq. (38). We thus have:

$$\frac{D}{Dt}b_{ij} + D_f(b) = -\frac{4}{3}K\Sigma_{ij} - \Omega_{ij} - Z_{ij} + B_{ij} - \pi_{ij} \tag{A.2}$$

where the (traceless) tensors $\boldsymbol{\Omega}$ and $\mathbf{Z}$ representing shear and vorticity are defined as:

$$\Omega_{ij} = b_{ik}\Sigma_{jk} + b_{jk}\Sigma_{ik} - \frac{2}{3}\delta_{ij}\Sigma_{k\ell}b_{k\ell} \tag{A 2}$$

$$Z_{ij} = b_{ik}V_{jk} + b_{jk}V_{ik} \tag{A.3}$$

where Σ_{ij} and V_{ij} are shear and vorticity:

$$\Sigma_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}), \qquad V_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i}) \tag{A.4}$$

The new tensor B_{ij} is given by

$$B_{ij} = g(\alpha_T L_{ij} - \alpha_c M_{ij}) \tag{A 5}$$

$$L_{ij} = \lambda_i J_j + \lambda_j J_i - \frac{2}{3}\delta_{ij}\lambda_k J_k \tag{A 6}$$

$$M_{ij} = \lambda_i \Phi_j + \lambda_j \Phi_i - \frac{2}{3}\delta_{ij}\lambda_k \Phi_k \tag{A 7}$$

We recall that

$$\lambda_i = -(g\bar{\rho})^{-1}\frac{\partial P}{\partial x_i} \tag{A 8}$$

Finally, we have to treat the pressure–velocity tensor. Following the procedure described in (Canuto 1994), we take

$$\bar{\rho}^{-1}\Pi_{ij} = 2\tau_{pv}^{-1}b_{ij} - \frac{4}{5}K\Sigma_{ij} - p_1\Omega_{ij} - p_2 Z_{ij} + (1-\beta_5)B_{ij} \tag{A.9}$$

where the numerical constants $p_{1,2}$ and β_5 are given in the text. The time scale τ_{pv} is discussed in Appendix B. Finally, Eq. (A.2) becomes

$$\frac{D}{Dt}b_{ij} + D_f(b) = -2\tau_{pv}^{-1}b_{ij} - \frac{8}{15}K\Sigma_{ij} - (1-p_1)\Omega_{ij} - (1-p_2)Z_{ij} + \beta_5 B_{ij} \tag{A.10}$$

Appendix B

The $(\tau_{pv}, \tau_{p\theta}, \tau_\theta)$ vs. τ relation is (Canuto and Dubovikov 1998):

$$\tau = 2K\epsilon^{-1}, \qquad \tau_{pv} = \frac{2}{5}\tau \tag{B.1}$$

for the T–field we have

$$\frac{\tau_{p\theta}}{\tau} = \frac{1}{4\pi^2}\mathrm{Pe}_\theta\,[1 + \frac{5}{4\pi^2}\mathrm{Pe}_\theta(1+\sigma_{t\theta}^{-1})]^{-1} \tag{B.2}$$

$$\frac{\tau_\theta}{\tau} = \frac{4}{7\pi^2}\mathrm{Pe}_\theta\,[1 + \frac{4}{7\pi^2}\mathrm{Pe}_\theta\sigma_{t\theta}^{-1}]^{-1} \tag{B.3}$$

For the C–field we have:

$$\frac{\tau_{pc}}{\tau} = \frac{1}{4\pi^2}\mathrm{Pe}_c\,[1 + \frac{5}{4\pi^2}\mathrm{Pe}_c(1+\sigma_{tc}^{-1})]^{-1} \tag{B 4}$$

$$\frac{\tau_c}{\tau} = \frac{4}{7\pi^2}\mathrm{Pe}_c[1 + \frac{4}{7\pi^2}\mathrm{Pe}_c\sigma_{tc}^{-1}]^{-1} \tag{B.5}$$

For the T–C correlation, we have:

$$\frac{\tau_{c\theta}}{\tau} = \frac{4}{7\pi^2}\mathrm{Pe}_\theta(1+\mathrm{Pe}_\theta/\mathrm{Pe}_c)^{-1}[1 + \frac{15}{7\pi^2}\mathrm{Pe}_\theta\sigma_{t\theta}^{-1}\,(1+\sigma_{t\theta}/\sigma_{tc})(1+\mathrm{Pe}_\theta/\mathrm{Pe}_c)^{-1}]^{-1} \tag{B 6}$$

The Peclet numbers $\mathrm{Pe}_{\theta,c}$ are defined as:

$$\mathrm{Pe}_{\theta,c} = \frac{4\pi^2}{125}\frac{K^2}{\epsilon}(\frac{1}{\chi_\theta}, \frac{1}{\chi_c}) \tag{B.7}$$

The turbulent Prandtl numbers $\sigma_{t\theta}$, σ_{tc} are themselves functions of the corresponding Pe's and satisfy the general equation. Calling $\sigma_t^{-1}\equiv\Sigma$, we have

$$\gamma_2\Sigma = 1 + \frac{2}{5}\pi^2\mathrm{Pe}^{-1}(\gamma_2-\sigma)[(1 + \frac{5}{2\pi^2}\mathrm{Pe}\,\frac{\gamma_1\Sigma+1}{\gamma_1+\sigma}\,)^{-\Gamma} -1] \tag{B.8}$$

with $2\gamma_1=(\gamma^2+4\gamma)^{\frac{1}{2}}-\gamma$, $\gamma_2=\gamma_1+\gamma$ and γ=0.3. The Prandtl number $\sigma=\nu/\chi$ is usually $O(10^{-8})$ and thus negligible.

The Peclet number Pe_c can safely be taken much larger than unity in which case both (B.4) and (B.5) become constant. When also $\mathrm{Pe}_\theta>>1$, we have

$$\sigma_t=0.72 \tag{B.9}$$

and thus:

$$\tau_{p\theta}/\tau=\tau_{pc}/\tau= \frac{1}{5}(1+\sigma_t^{-1})^{-1},\ \tau_\theta/\tau=\tau_c/\tau = \sigma_t,\ \tau_{c\theta}/\tau= \frac{2}{15}\sigma_t \tag{B.10}$$

or

$$\tau_{p\theta}/\tau=\tau_{pc}/\tau =0.0837$$

$$\tau_\theta/\tau = \tau_c/\tau = 0\ 72$$

$$\tau_{c\theta}/\tau = 0.096$$

These values in turn imply that Eq (84a) becomes:

$$p_1 = 0.832, \quad p_2 = 0.545, \tag{B 11}$$

$$p_3 = 0.2093, \quad p_4 = 0.0323$$

$$p_5 = 0.0155, \quad p_6 = 0.2422$$

$$p_7 = 0.4799, \quad p_8 = 0.2093$$

$$p_9 = 0.8721, \quad p_{10} = 0\ 0155$$

We also have

$$p_{11} = 0.1163$$

Thus:

$$p_{1m} = 0.168, \quad p_{2m} = 0.455 \tag{B.12}$$

$$a_1 = 1.0494, \quad a_2 = 0.9239 \tag{B 13}$$

$$a_3 = 0.0163, \quad a_4 = -10.4205, \quad a_5 = -1.3656$$

$$b_1 = -0.1008, \quad b_2 = -0.1163$$

$$b_3 = 0.5702, \quad b_4 = -0.9689 \tag{B.14}$$

$$b_5 = -7.2674, \quad b_6 = -0.0155$$

$$b_7 = -0.7558$$

$$d_1 = 0.1111, \quad d_2 = 0\ 1042$$

$$d_3 = 0.0017, \quad d_4 = -0.3494 \tag{B.15}$$

$$d_5 = -0.4353, \quad d_6 = -0.0572$$

$$d_7 = -0.0007, \quad d_8 = -1.0938$$

$$d_9 = -0.1435, \quad d_{10} = 6.2271$$

$$d_{11} = 4.0950, \quad d_{12} = 0.1034$$

$$d_{13} = 1.1857, \quad d_{14} = -30.5038$$

$$d_{15} = -4.0196$$

The quantities n_0 and c_0 entering (86a,b), as well as (90b,c), are then: (B 16)

$$n_0 = 0.0691, \quad c_0 = 0.5184 \tag{B.17}$$

Appendix C

The functions a,b,c and d entering Eqs.(88a)–(88e) are given by:

$$
\begin{aligned}
a_1 &= p_{11}[12p_9+8p_6-30p_6p_8-5p_6(p_{1m}+3p_{2m})],\\
a_2 &= -5[(p_4p_9+p_6p_{11}-2p_4p_6p_7)(p_{1m}+3p_{2m})+\\
&+8(p_4p_9+2p_6p_{11}-2p_5p_6)+12(p_{11}p_9+p_{10}p_{11}-p_5p_6p_7)\\
&-30(p_3p_4p_9+p_6p_8p_{11}-p_5p_6p_8-p_3p_5p_6)\\
a_3 &= p_{10}[8p_4+12p_{11}-30p_3p_4-5p_4(p_{1m}+3p_{2m})]\\
a_4 &= -p_6(8-30p_8-5p_{1m}-15p_{2m})-12(p_9+p_{11})\\
a_5 &= -p_4(8-30p_3-5p_{1m}-15p_{2m})-12(p_{10}+p_{11})
\end{aligned}
\tag{C 1}
$$

$$
\begin{aligned}
&b_1 = p_4p_7-p_{11},\ b_2=-p_{11},\ b_3 = 15p_{2m}^2+2p_{1m}-5p_{1m}^2-6p_{2m}\\
&b_4 = -30p_4,\ b_5 = -30p_6,\ b_6 = -p_{10}.\ b_7=p_6p_7-p_9
\end{aligned}
\tag{C 2}
$$

$$
\begin{aligned}
d_1 &= p_{11}[p_{2m}^2(p_6+6p_9)+2(p_{1m}-3p_{2m})p_6p_8-p_{1m}^2(p_6+2p_9)]\\
d_2 &= (p_{1m}^2-p_{2m}^2)(2p_4p_7p_6-p_6p_{11}-p_4p_9)+\\
&+2(p_{1m}^2-3p_{2m}^2)(p_4p_6p_7^2-p_{11}p_9-p_{10}p_{11})+\\
&+2(p_{1m}-3p_{2m})(p_3p_4p_9+p_6p_8p_{11}-p_4p_6p_7p_8-p_3p_4p_6p_7)\\
d_3 &= p_{10}[p_{2m}^2(p_4+6p_{11})+2(p_{1m}-3p_{2m})p_3p_4-p_{1m}^2(p_4+2p_{11})]\\
d_4 &= -4p_6p_{11}(2p_6+3p_9)\\
d_5 &= 4p_4p_7p_6^2(4+3p_7)-4p_4p_9(3p_{11}+2p_6)-4p_6p_{11}(3p_9+3p_{10}+2p_4+2p_6)\\
d_6 &= 4p_4^2p_6p_7(4+3p_7)-4p_4p_9(2p_4+3p_{11})-8p_4p_6(p_{10}+p_{11})-12p_{10}p_{11}(p_4+p_6)\\
d_7 &= -4p_4p_{10}(2p_4+3p_{11})\\
d_8 &= p_{1m}^2(2p_9+2p_{11}+p_6)-p_{2m}^2(6p_9+6p_{11}+p_6)-2p_6p_8(p_{1m}-3p_{2m})\\
d_9 &= p_{1m}^2(2p_{11}+2p_{10}+p_4)-p_{2m}^2(6p_{11}+6p_{10}+p_4)-2p_3p_4(p_{1m}-3p_{2m})\\
d_{10} &= 8p_6^2+4p_6(3p_9+7p_{11})+24p_9p_{11}\\
d_{11} &= -8p_4p_6p_7(4+3p_7)+4p_4(4p_6+7p_9+3p_{11})+4p_6(3p_{10}+7p_{11})+24p_{11}(p_9+p_{10})\\
d_{12} &= 4p_{10}(7p_4+6p_{11})+4p_4(2p_4+3p_{11}),\quad d_{13} = 6p_{2m}^2-2p_{1m}^2\\
d_{14} &= -24p_9-24p_{11}-28p_6,\quad d_{15} = -24p_{11}-24p_{10}-28p_4
\end{aligned}
\tag{C.3}
$$

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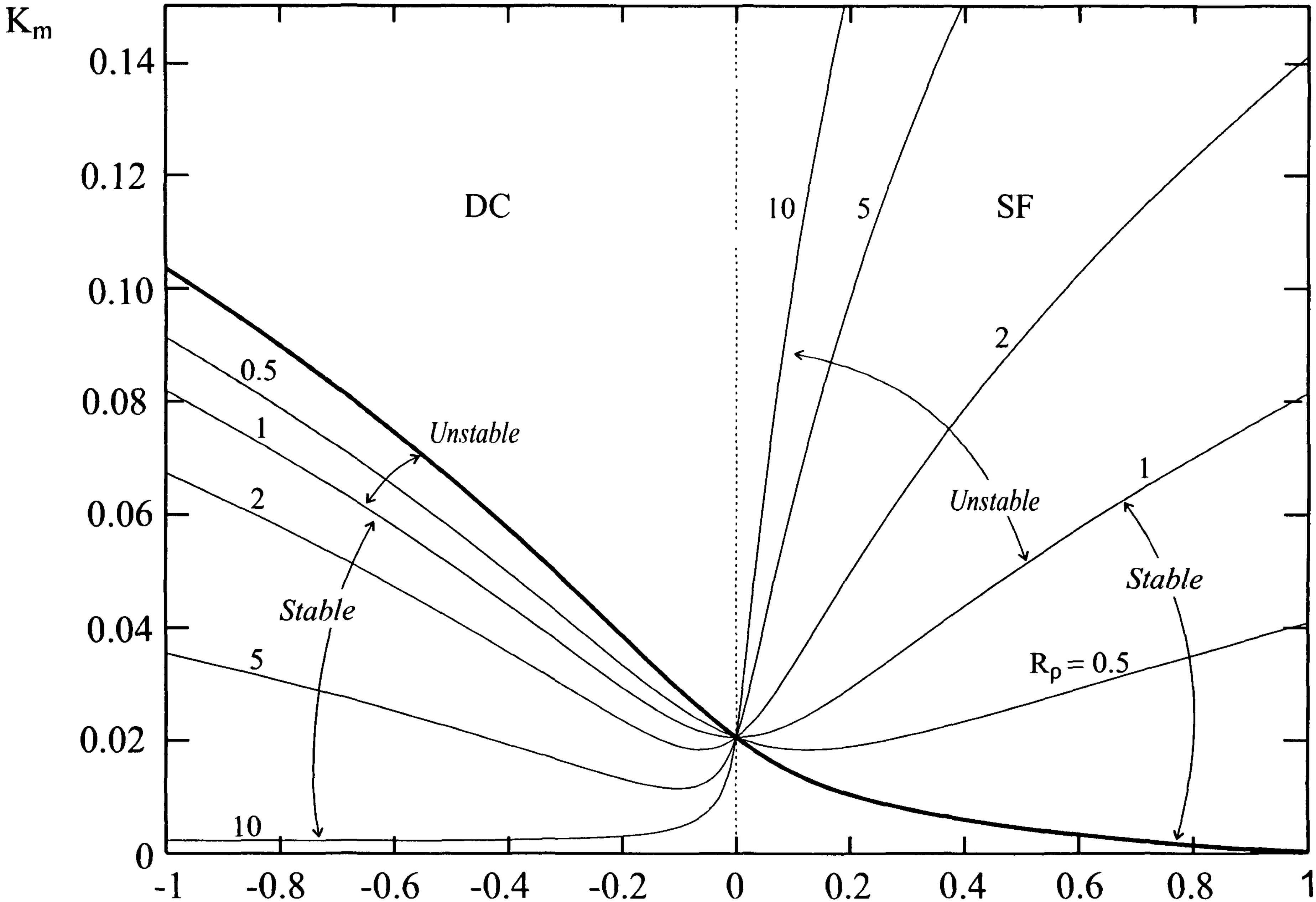
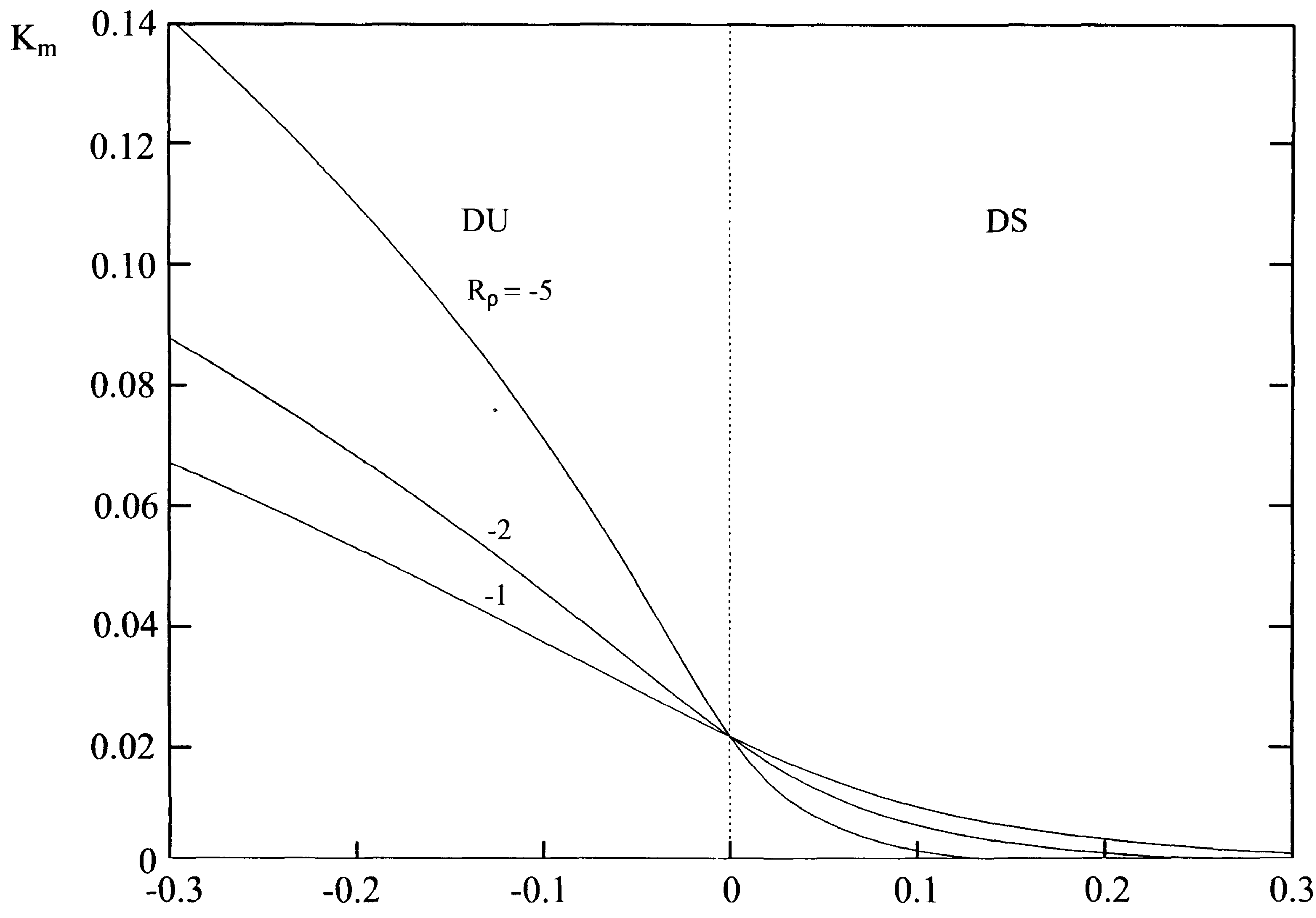


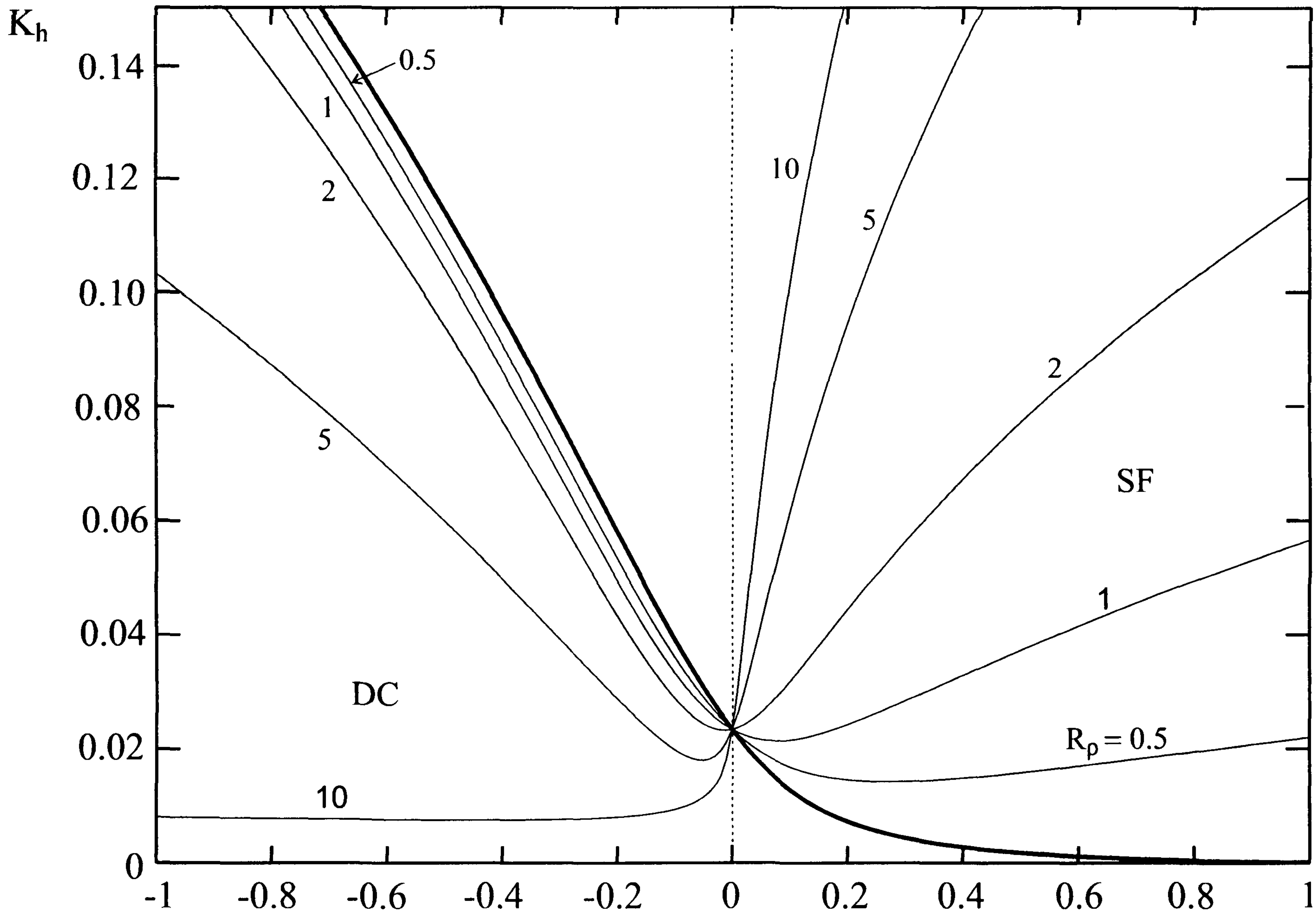
K_m
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri
DC
SF
10
5
2
1
$R_\rho = 0.5$
0.5
Unstable
Stable
Unstable
Stable

Fig.1a



K_m
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
Ri
DU
DS
$R_\rho = -5$
-2
-1

Fig.1b



K_h
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri
0.5
1
2
5
10
DC
SF
10
5
2
1
R_ρ = 0.5

Fig.2a

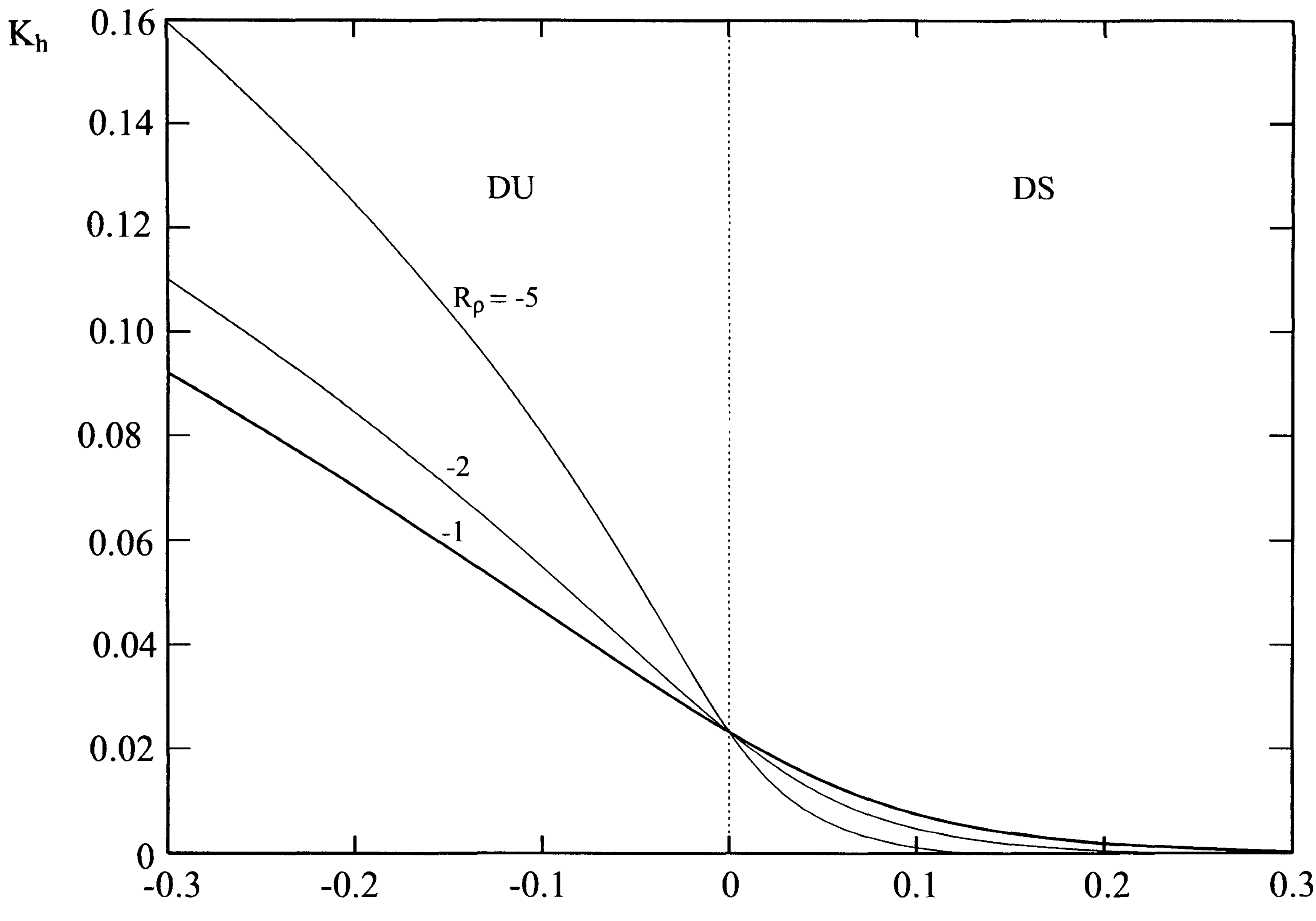
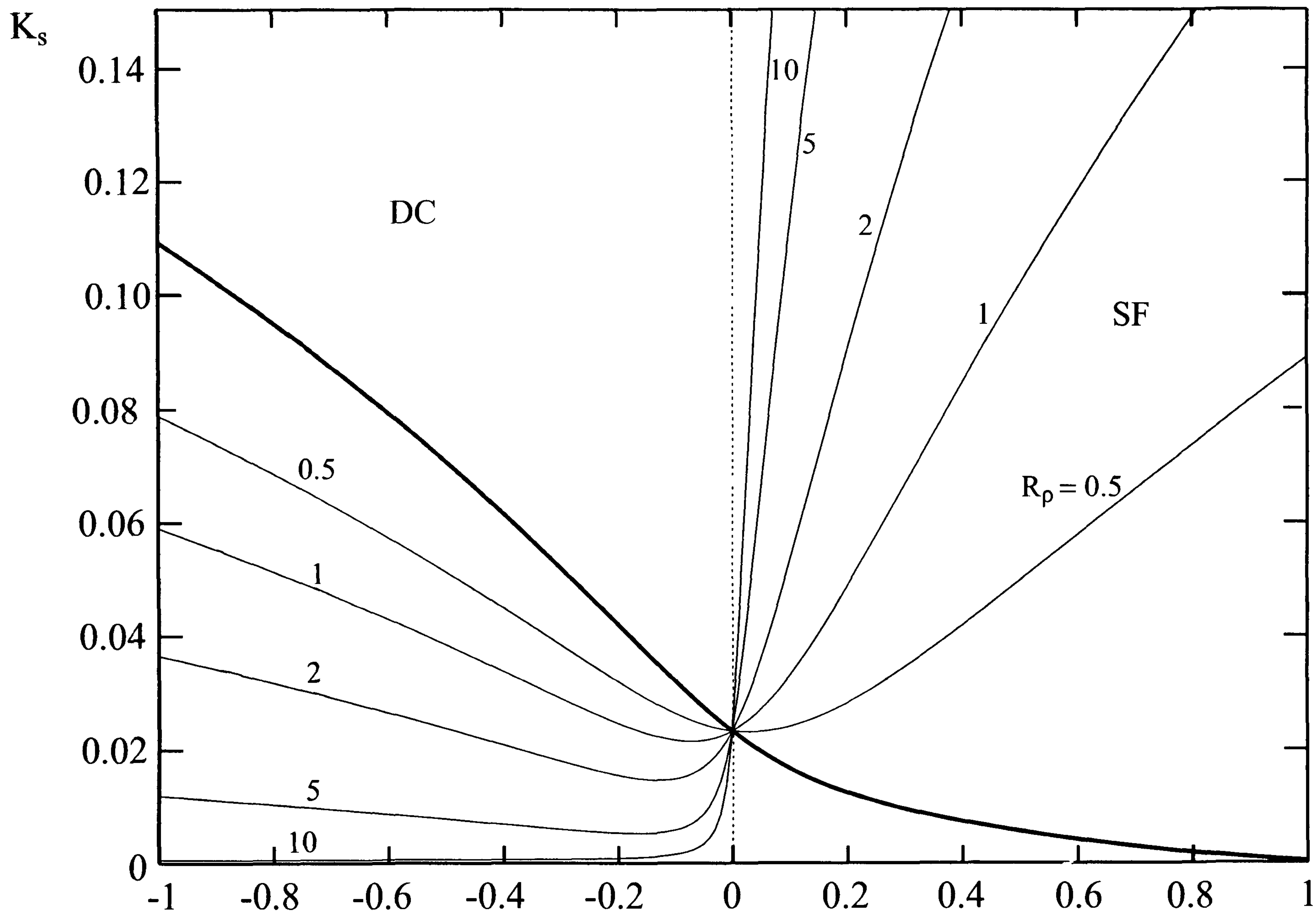

K_h
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
Ri
DU
DS
R_ρ = -5
-2
-1

Fig.2b



K_s
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
DC
SF
10
5
2
1
$R_\rho = 0.5$
0.5
1
2
5
10
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri

Fig.3a

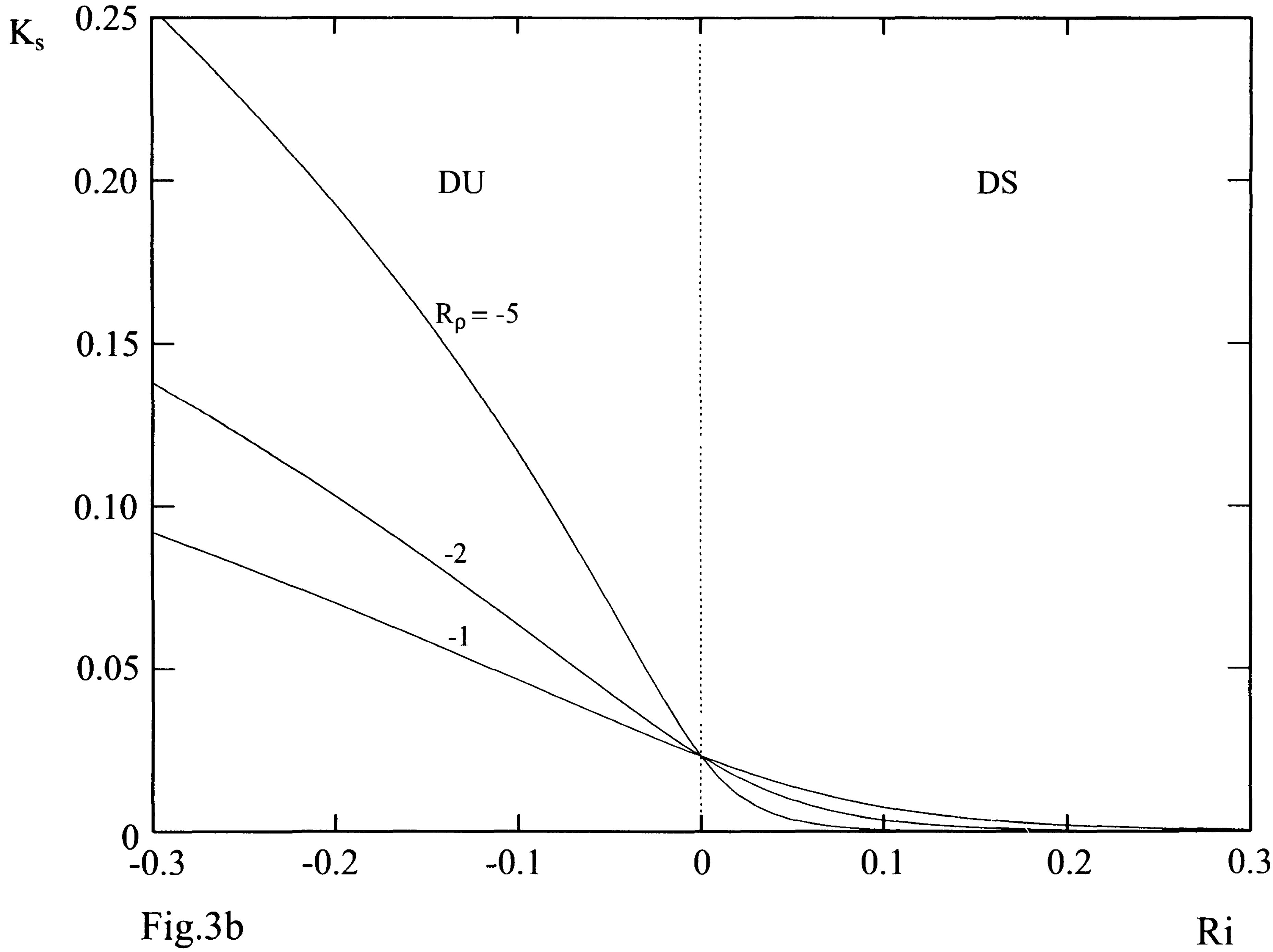


Fig.3b

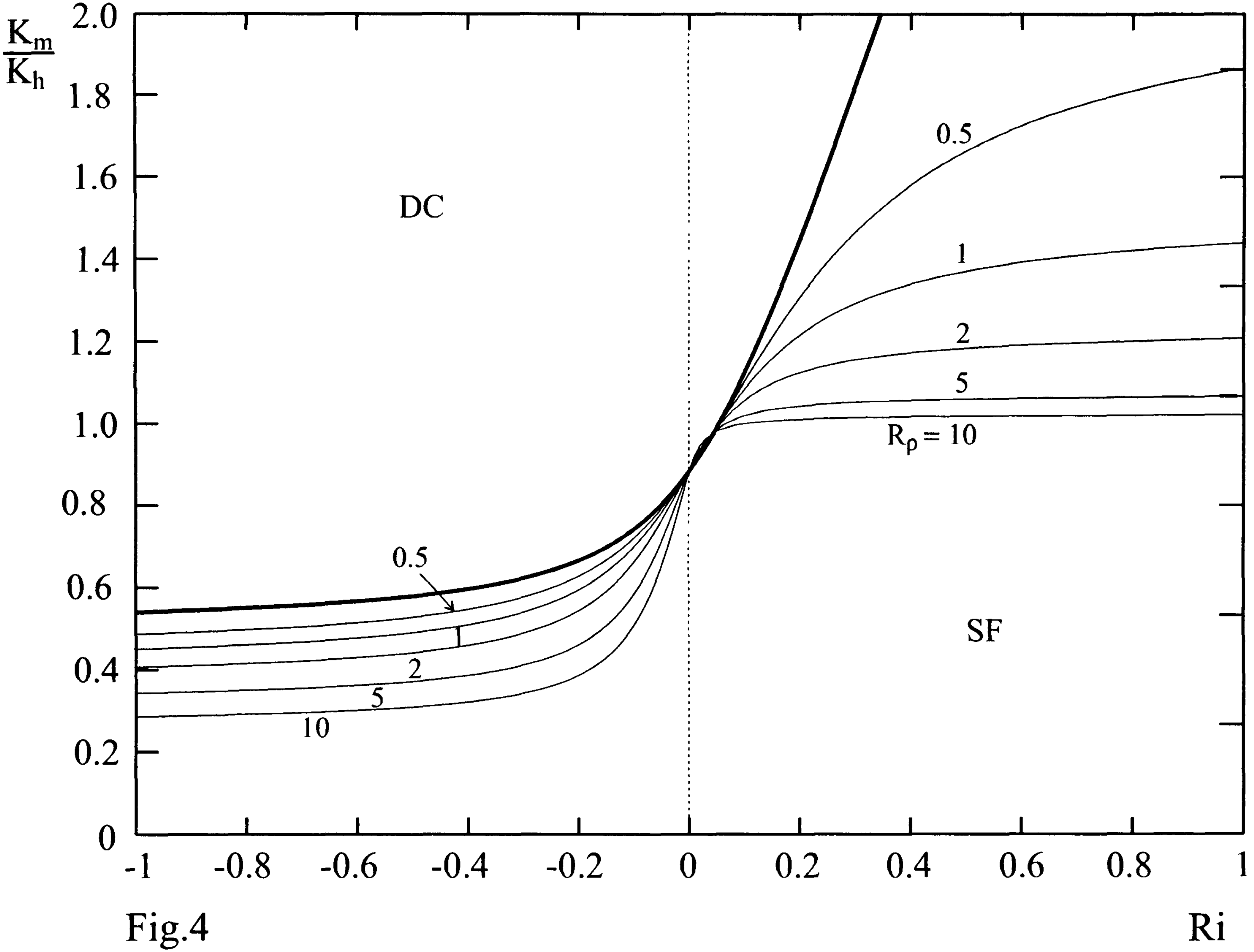
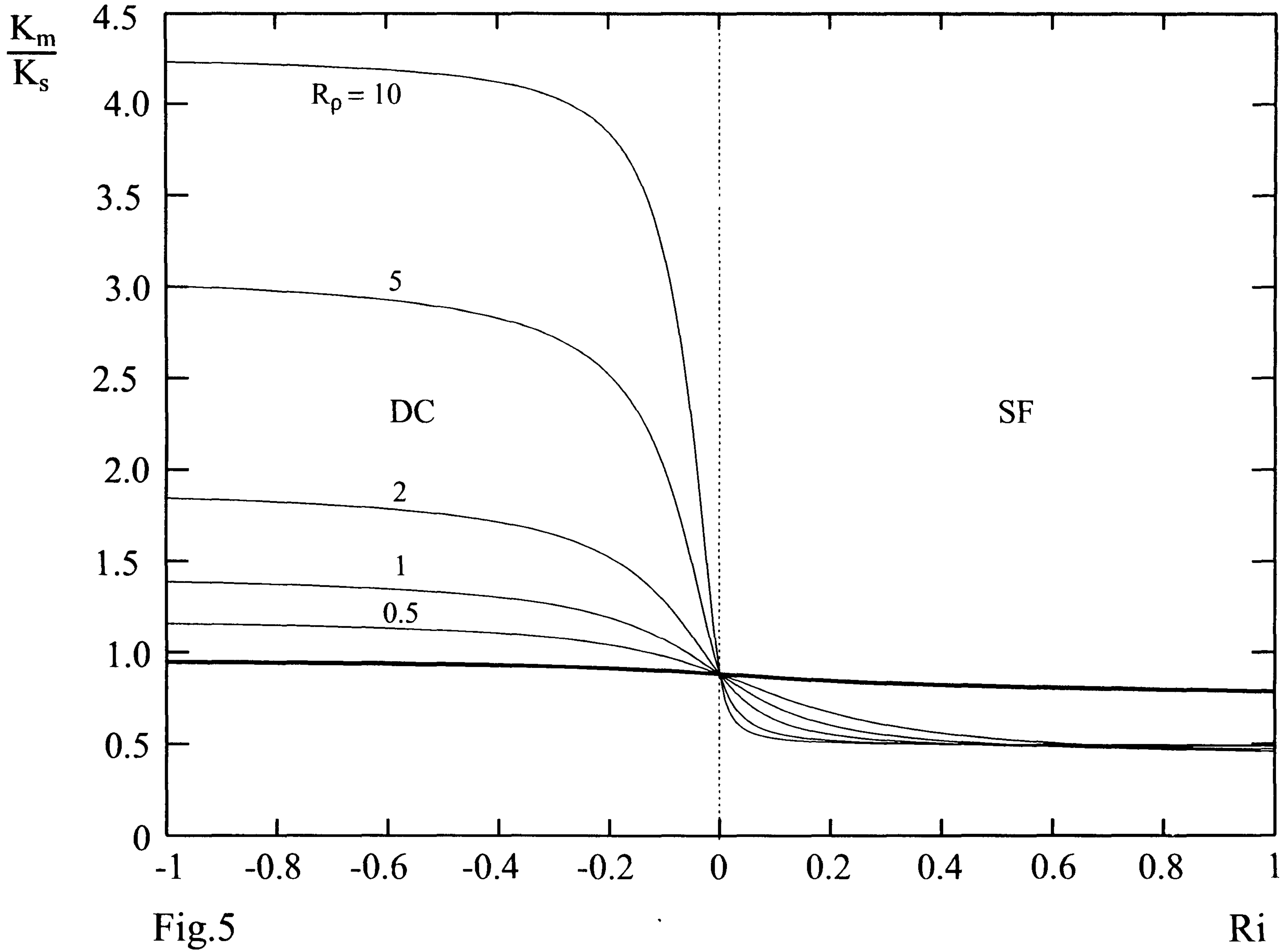
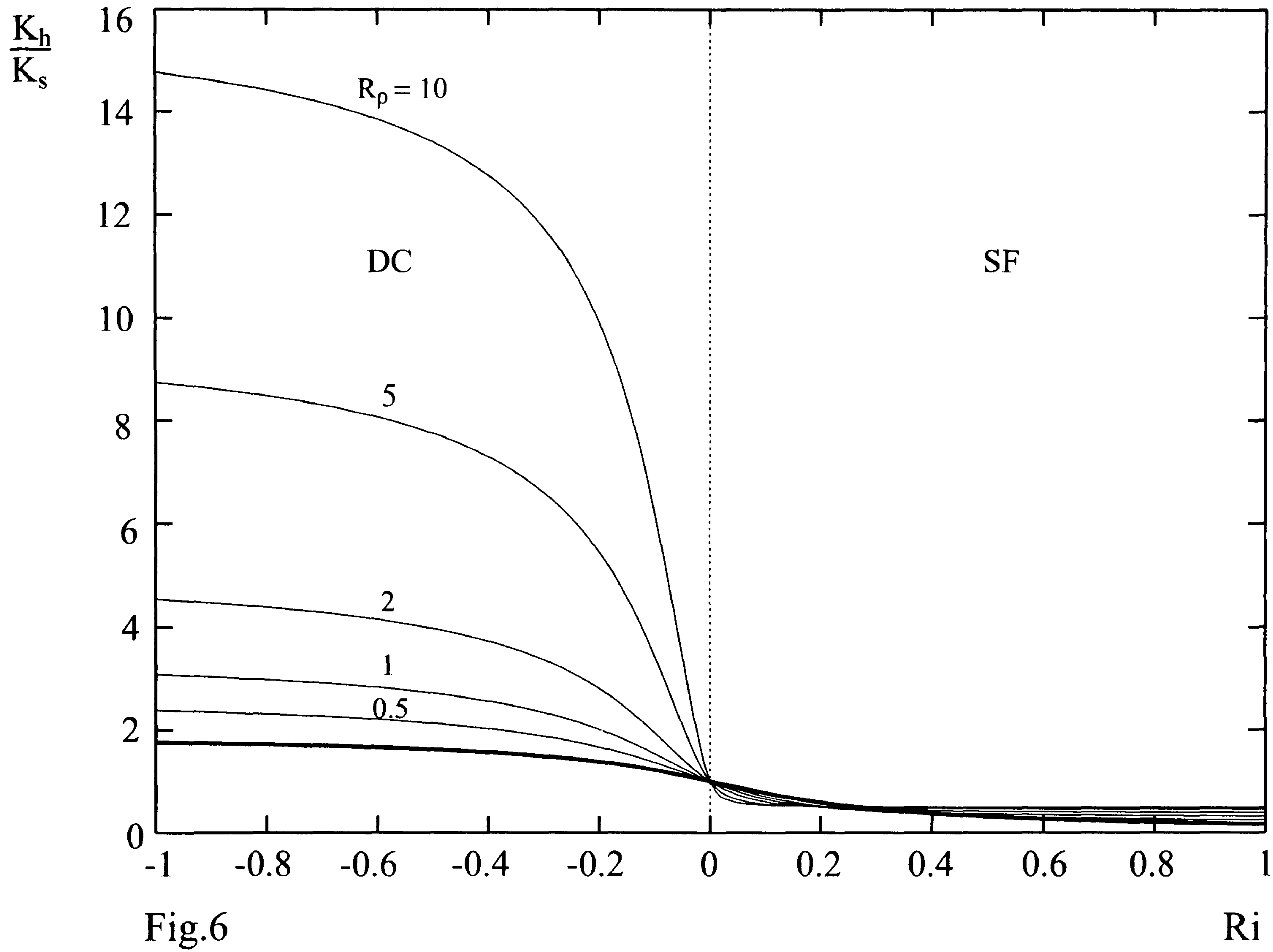

$\frac{K_m}{K_h}$
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri
DC
SF
0.5
1
2
5
$R_\rho = 10$
0.5
1
2
5
10

Fig.4



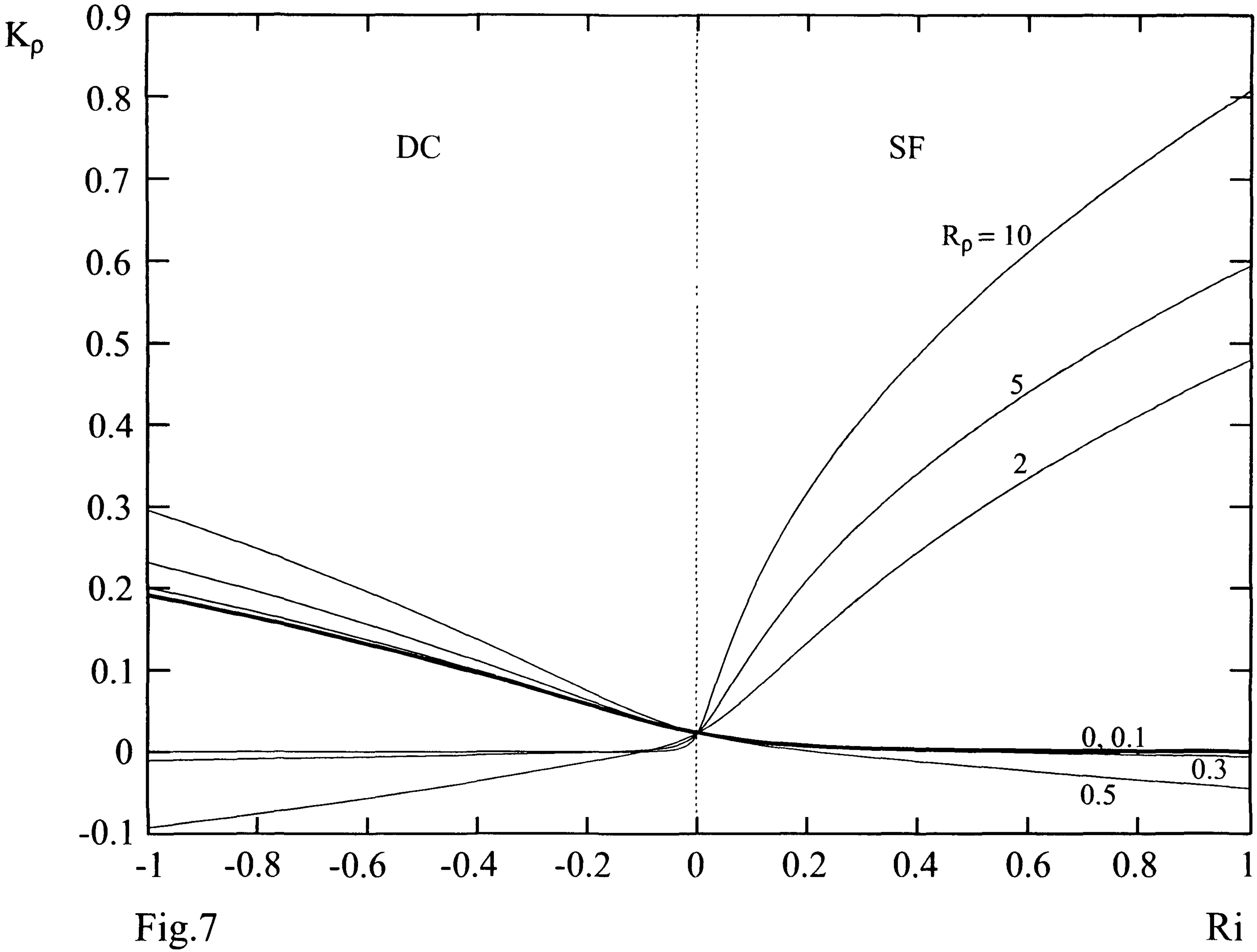
$\frac{K_m}{K_s}$
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
$R_\rho = 10$
5
2
1
0.5
DC
SF
Ri

Fig.5



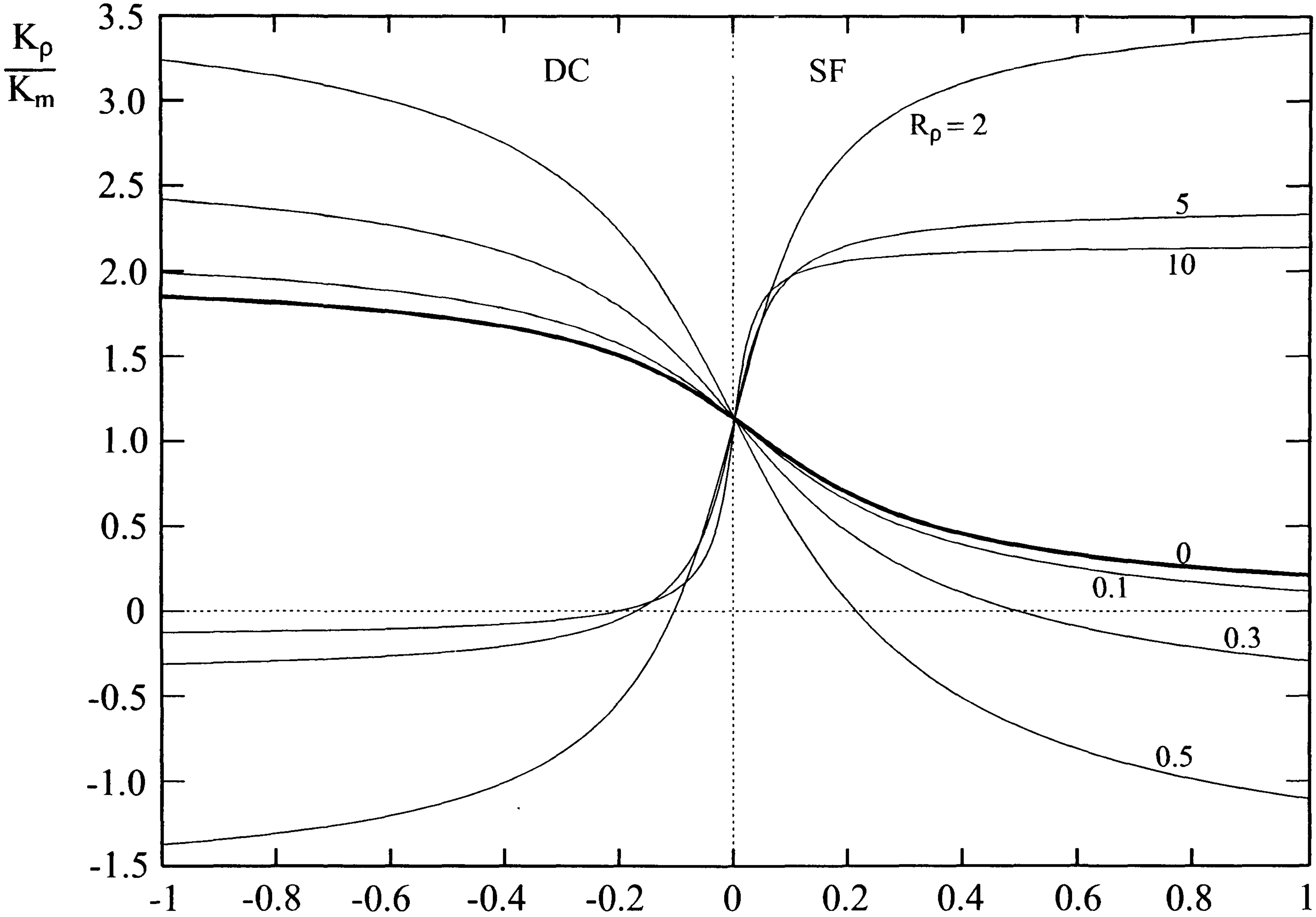
$\frac{K_h}{K_s}$
$R_\rho = 10$
DC
SF
5
2
1
0.5
16
14
12
10
8
6
4
2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri

Fig.6



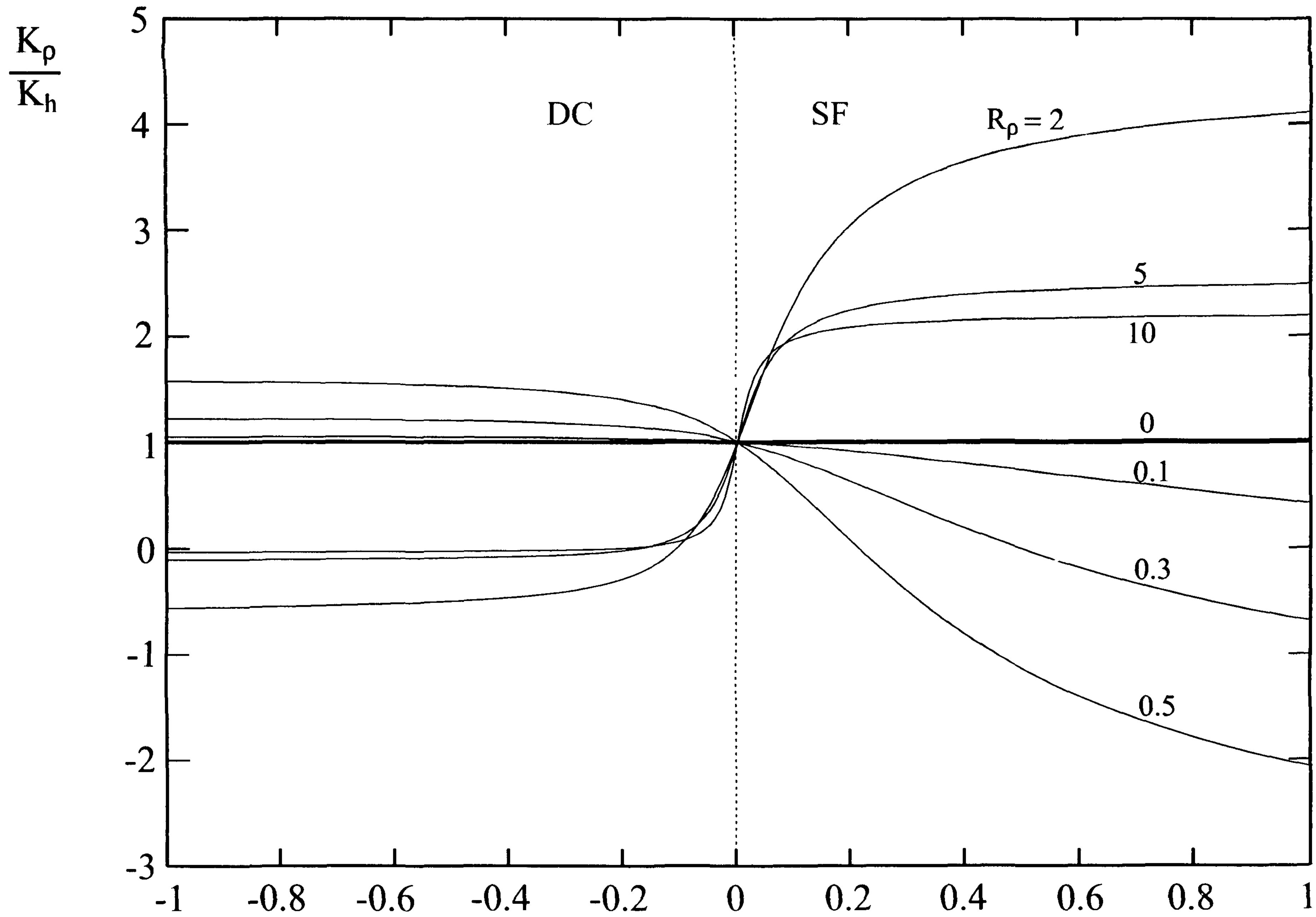
K_ρ
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
DC
SF
R_ρ = 10
5
2
0, 0.1
0.3
0.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri

Fig.7



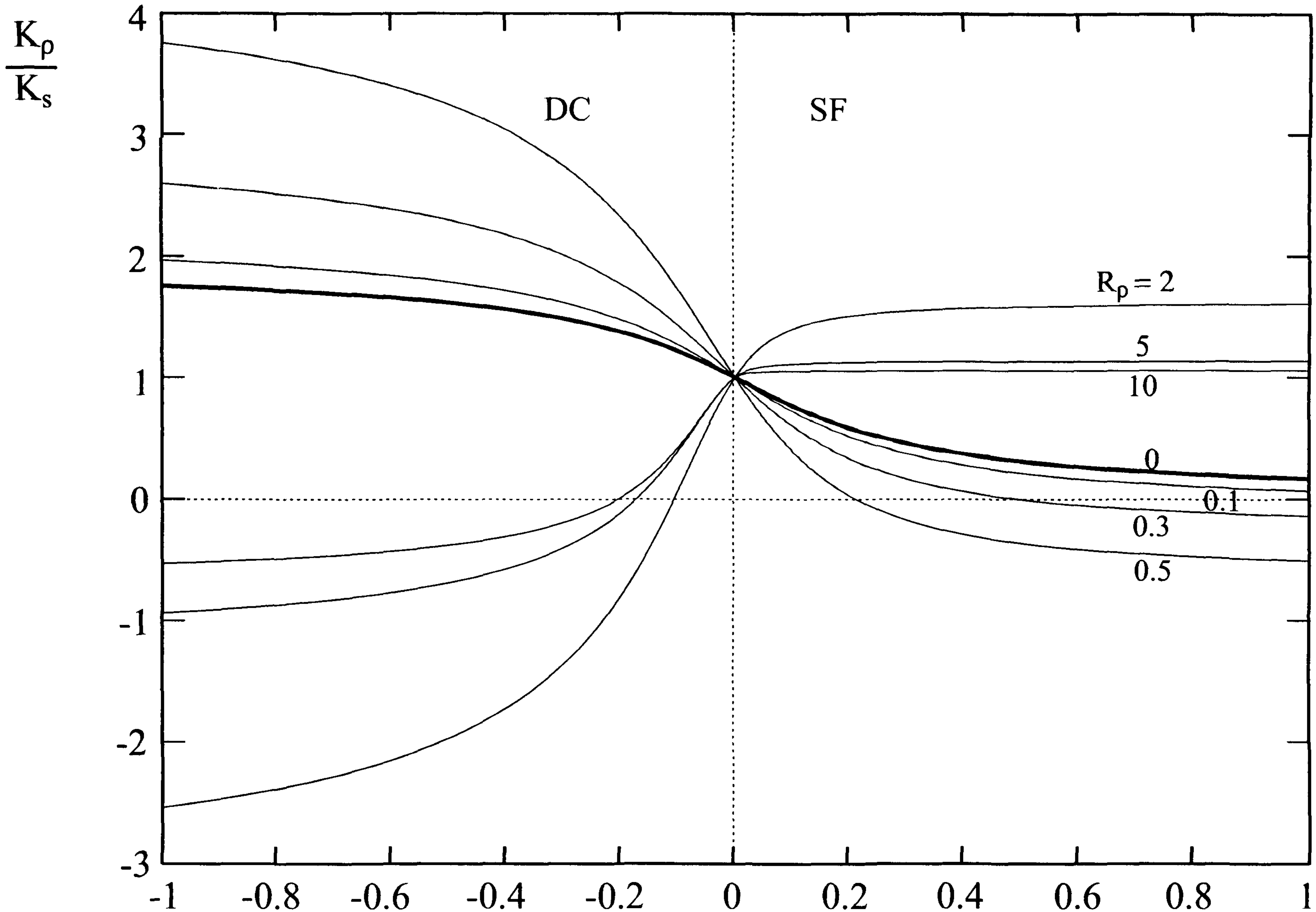
$\frac{K_\rho}{K_m}$
DC
SF
$R_\rho = 2$
5
10
0
0.1
0.3
0.5
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri

Fig.8



$\frac{K_\rho}{K_h}$
DC
SF
$R_\rho = 2$
5
10
0
0.1
0.3
0.5
5
4
3
2
1
0
-1
-2
-3
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri

Fig.9



$\frac{K_\rho}{K_s}$
DC
SF
$R_\rho = 2$
5
10
0
0.1
0.3
0.5
4
3
2
1
0
-1
-2
-3
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Ri

Fig.10

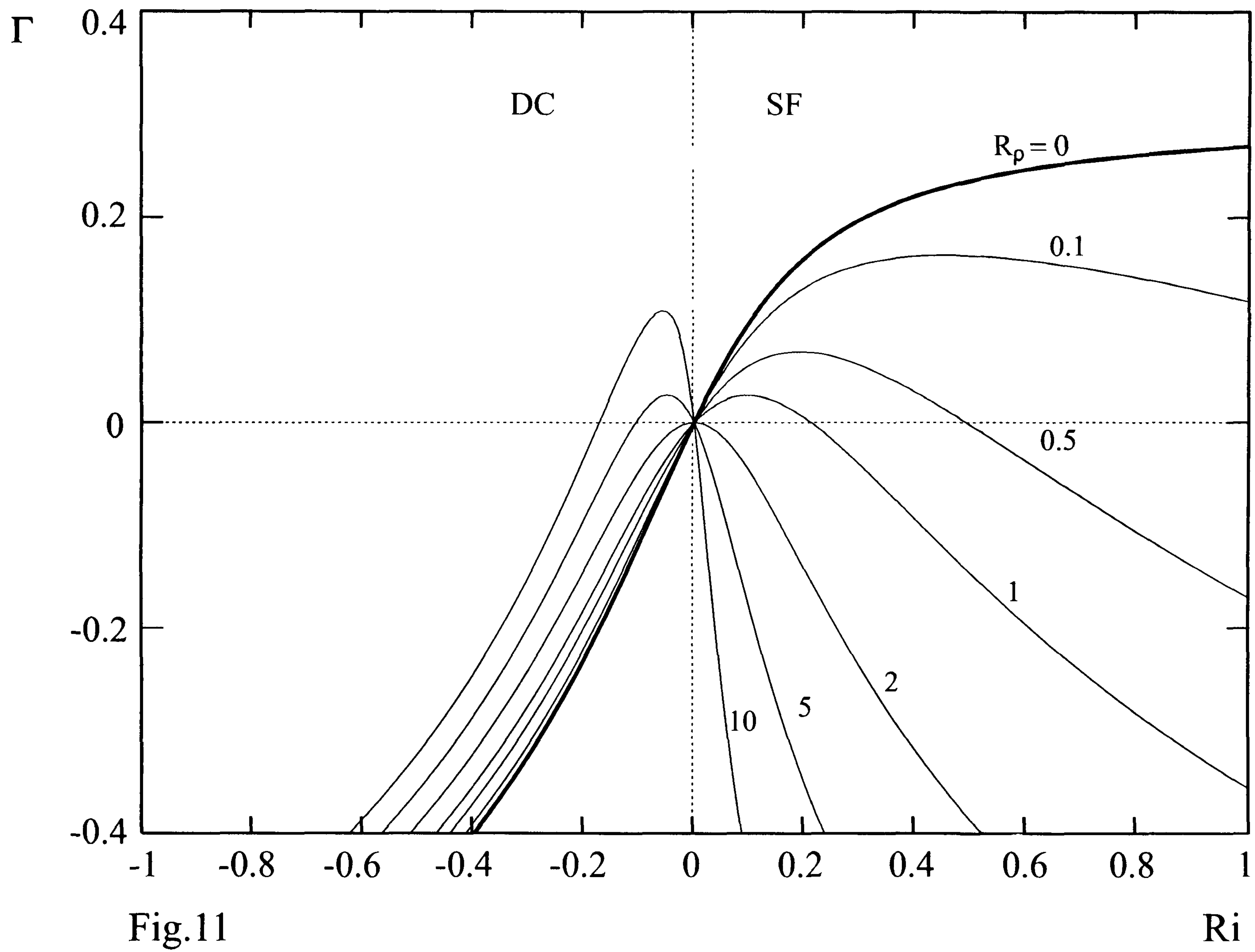


Fig.11

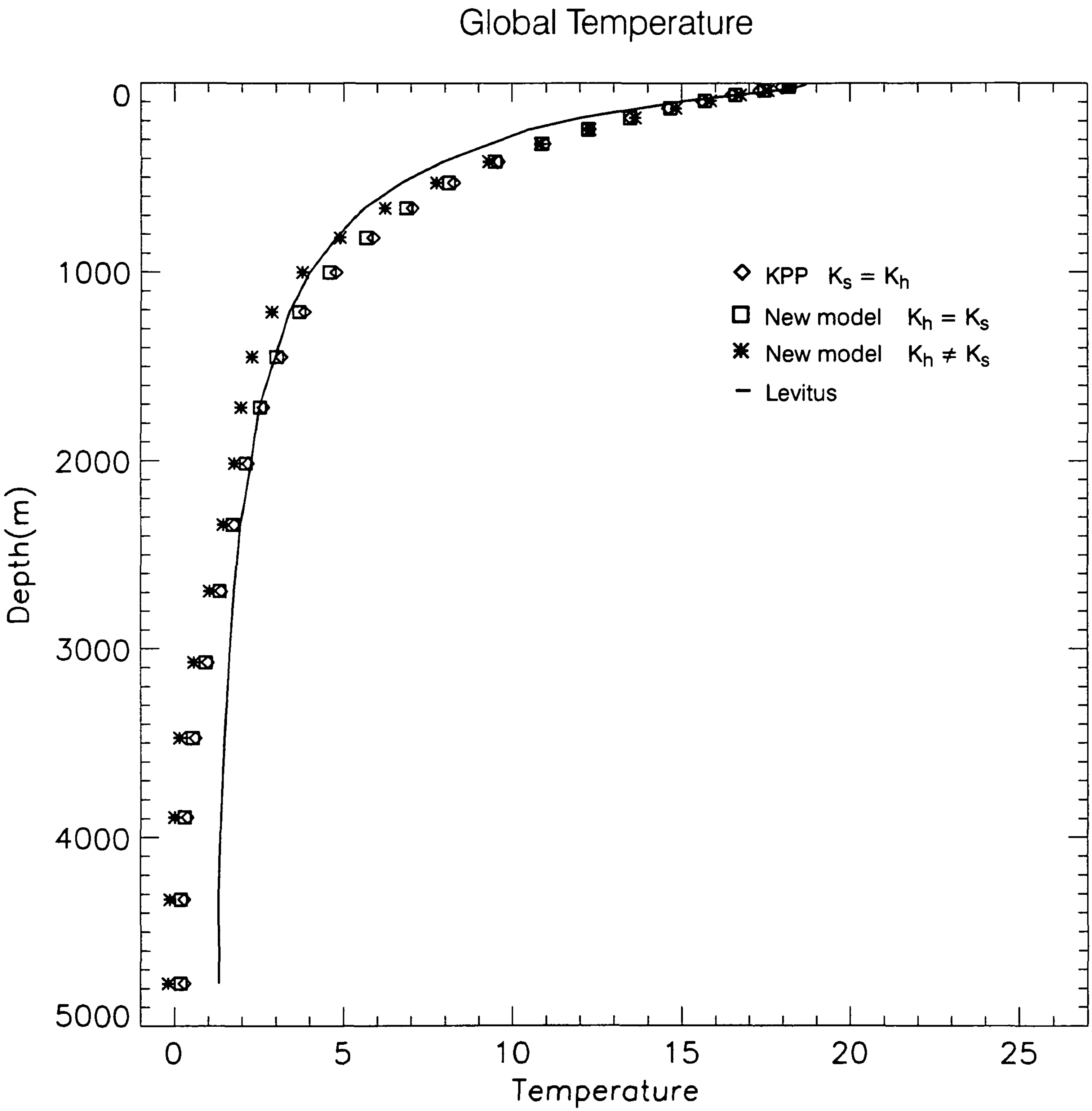

Global Temperature
KPP $K_s = K_h$
New model $K_h = K_s$
New model $K_h \neq K_s$
Levitus
Depth(m)
Temperature
0
1000
2000
3000
4000
5000
0
5
10
15
20
25

Fig. 12

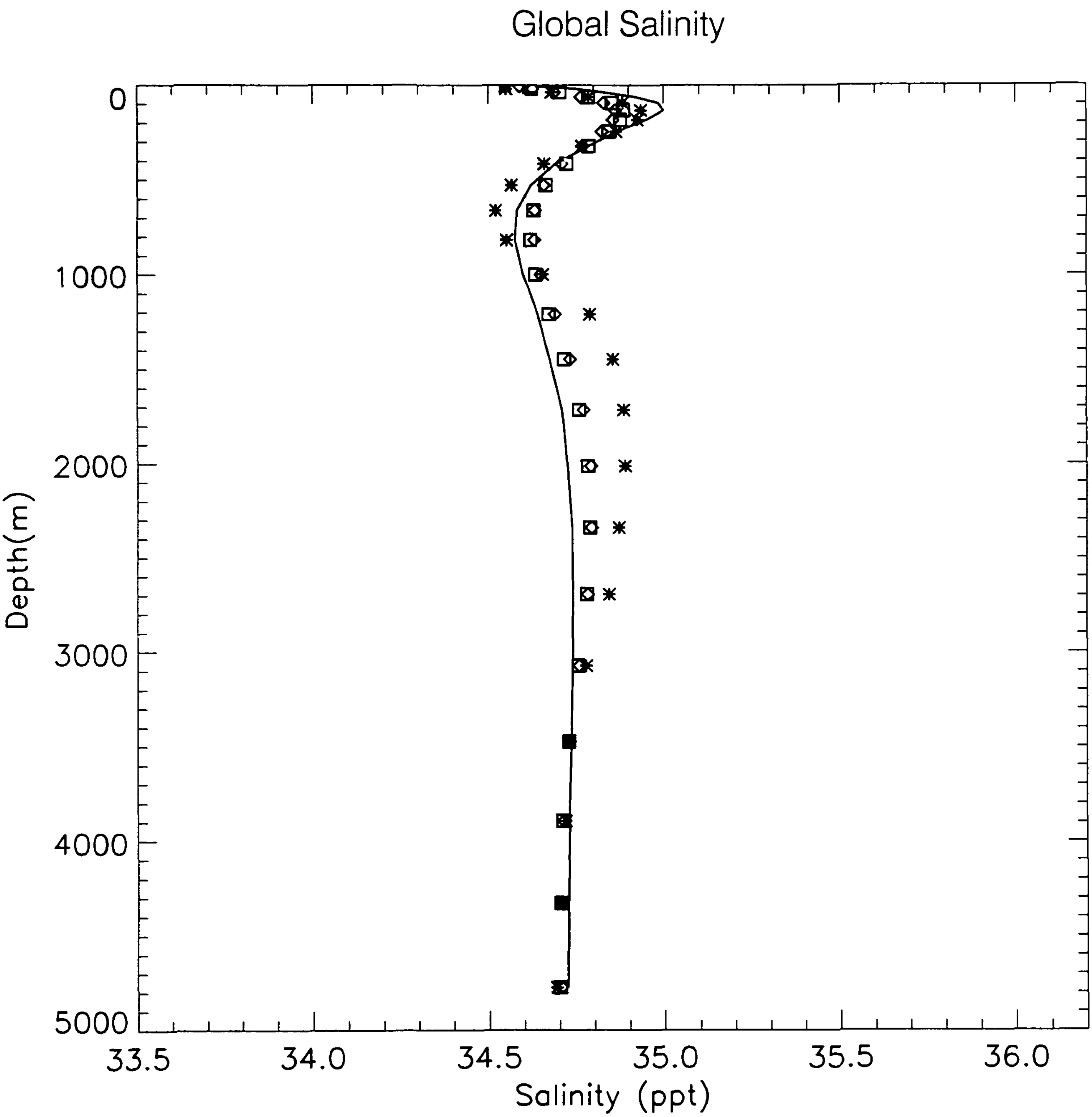

Global Salinity
Depth(m)
0
1000
2000
3000
4000
5000
33.5
34.0
34.5
35.0
35.5
36.0
Salinity (ppt)

Fig. 13

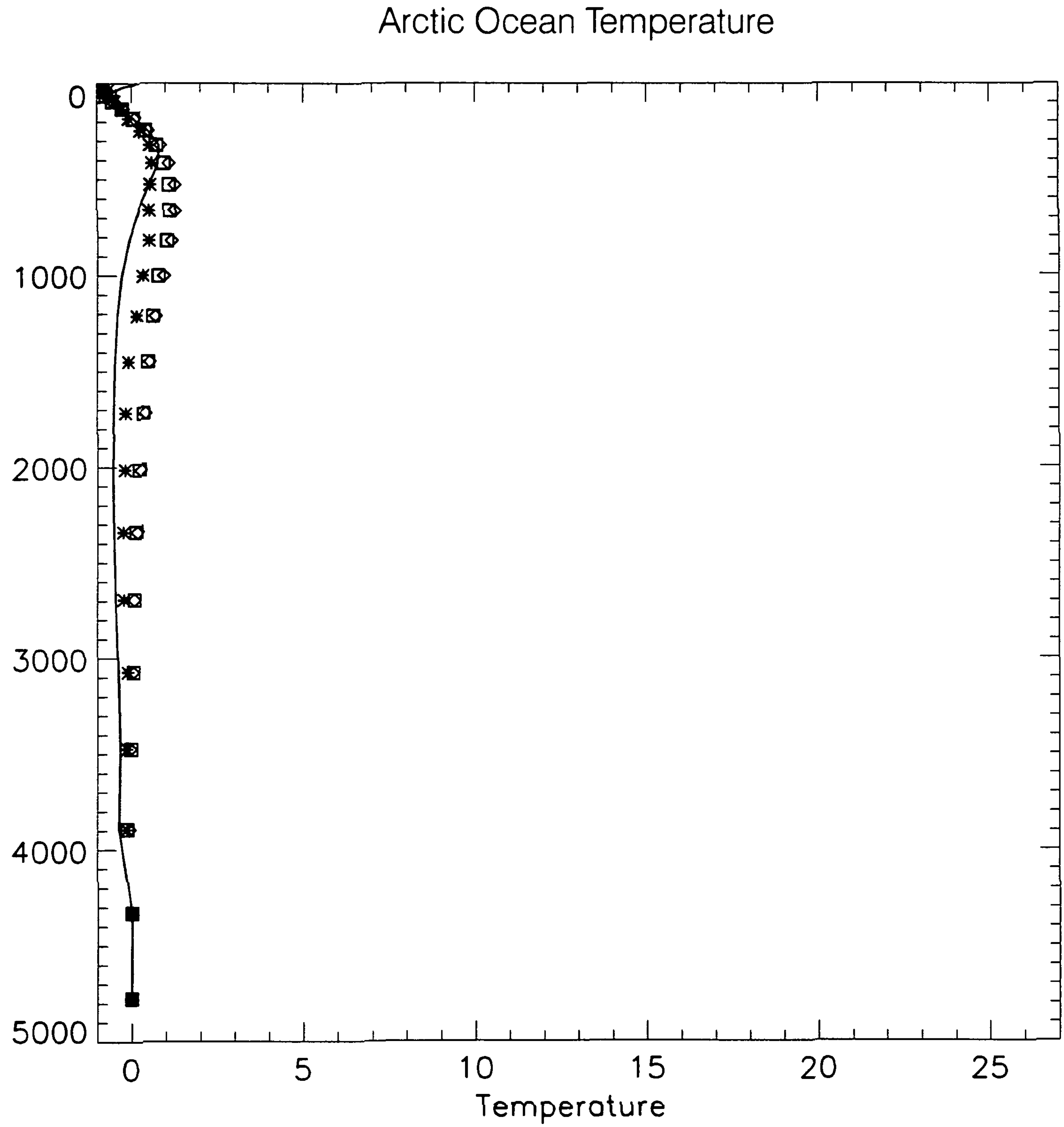

Arctic Ocean Temperature
0
1000
2000
3000
4000
5000
0
5
10
15
20
25
Temperature

Fig. 14

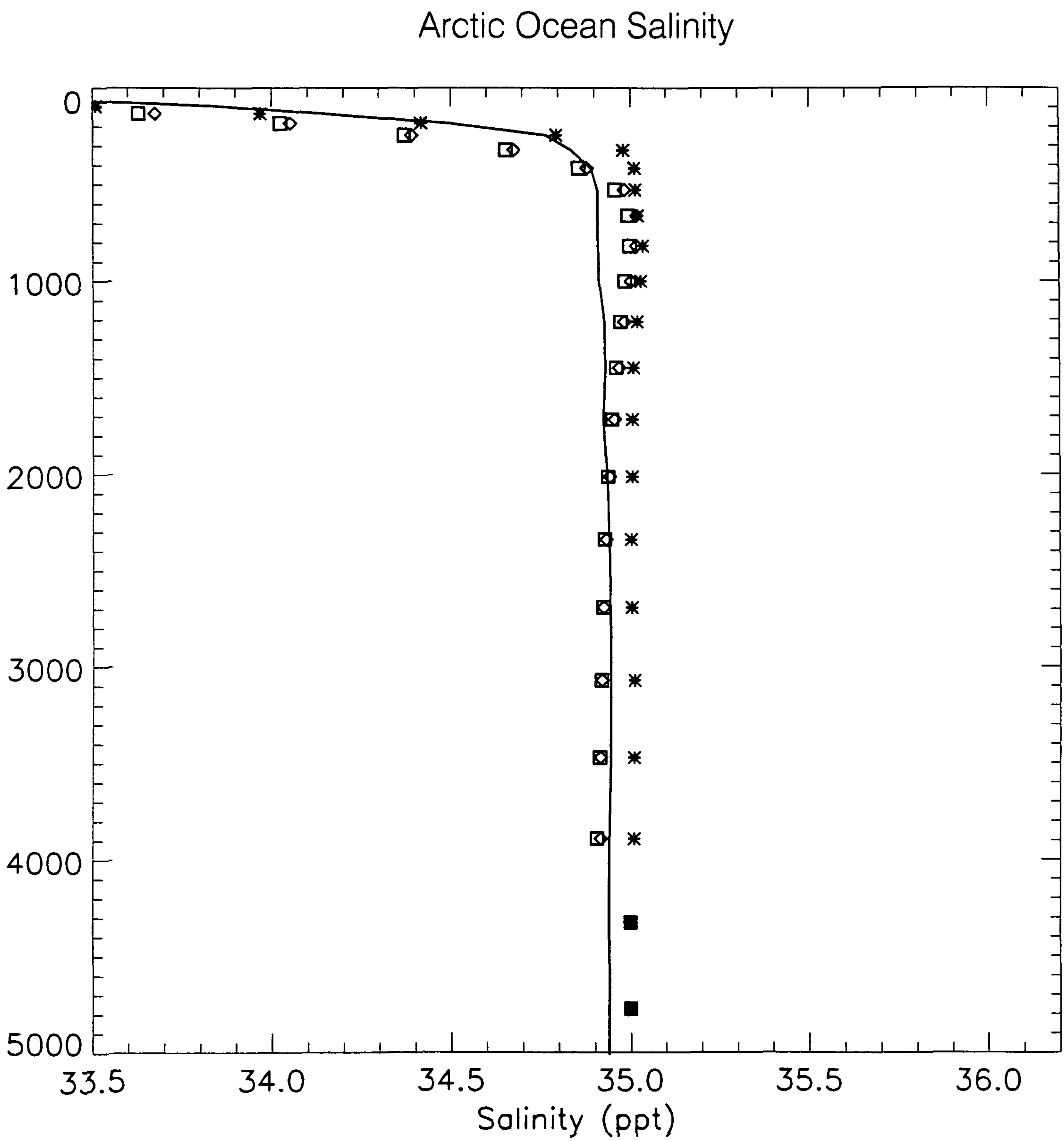

Arctic Ocean Salinity
0
1000
2000
3000
4000
5000
33.5
34.0
34.5
35.0
35.5
36.0
Salinity (ppt)

Fig. 15

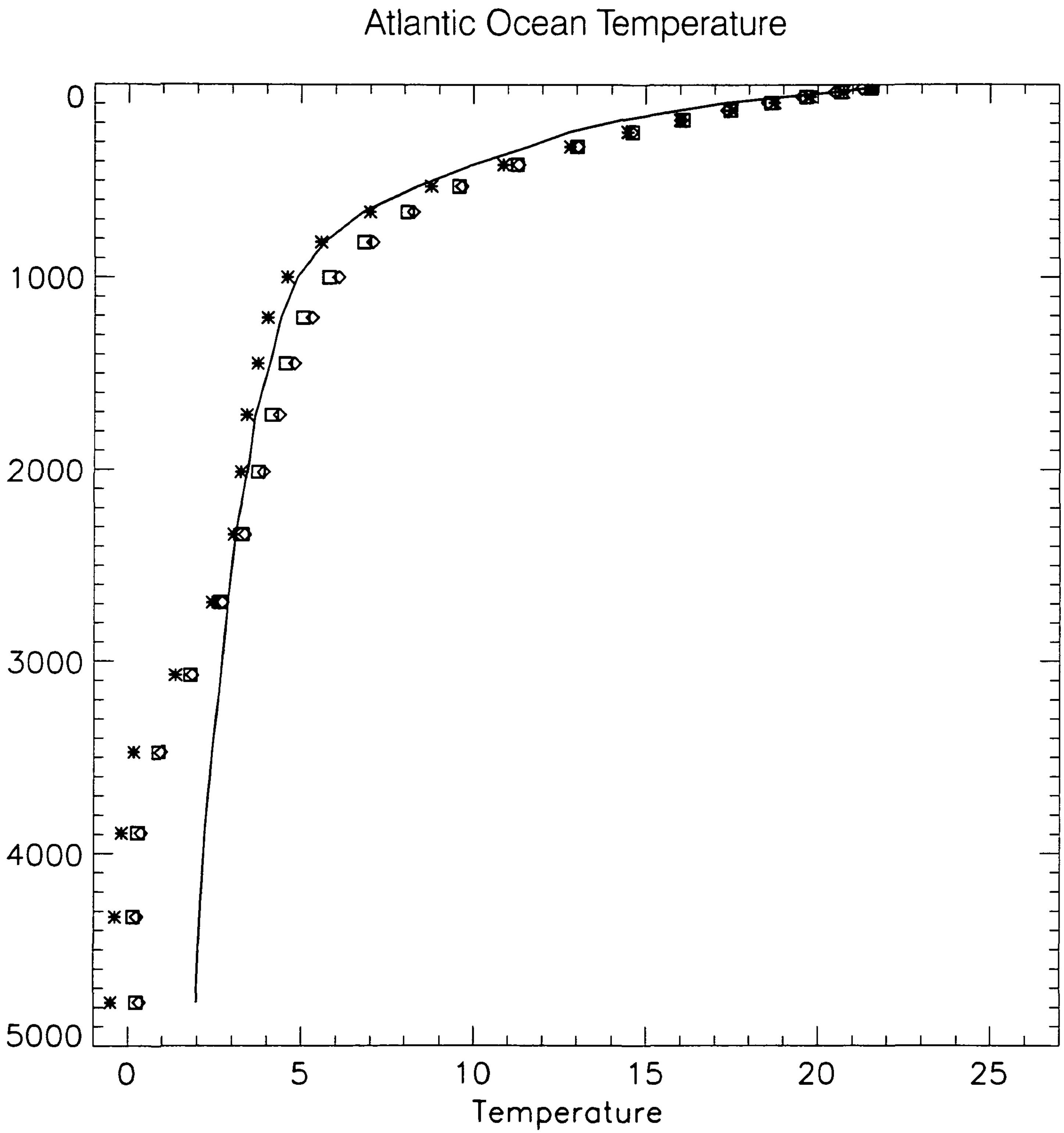

Atlantic Ocean Temperature
0
1000
2000
3000
4000
5000
0
5
10
15
20
25
Temperature

Fig. 16

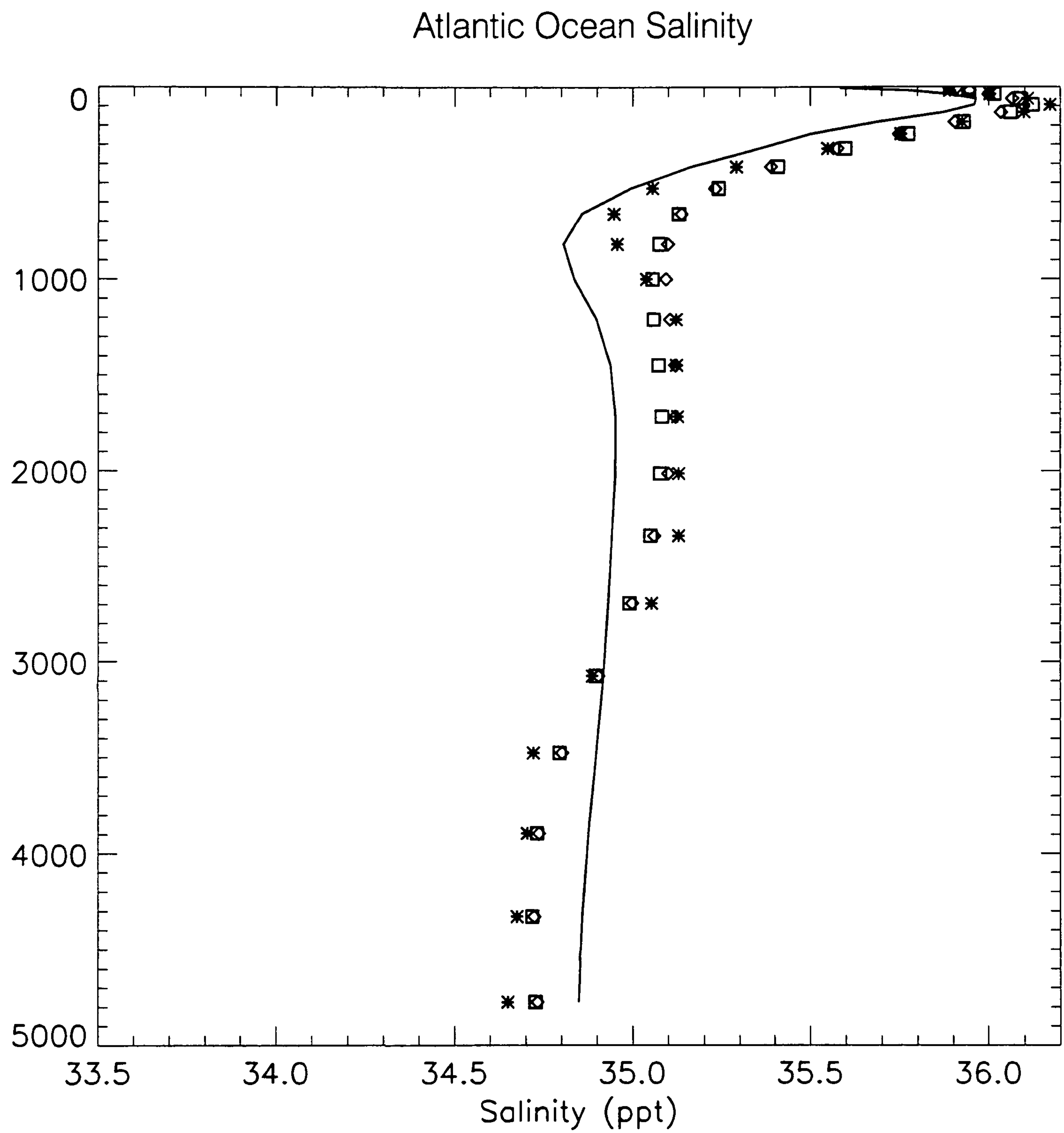

Atlantic Ocean Salinity
0
1000
2000
3000
4000
5000
33.5
34.0
34.5
35.0
35.5
36.0
Salinity (ppt)

Fig. 17

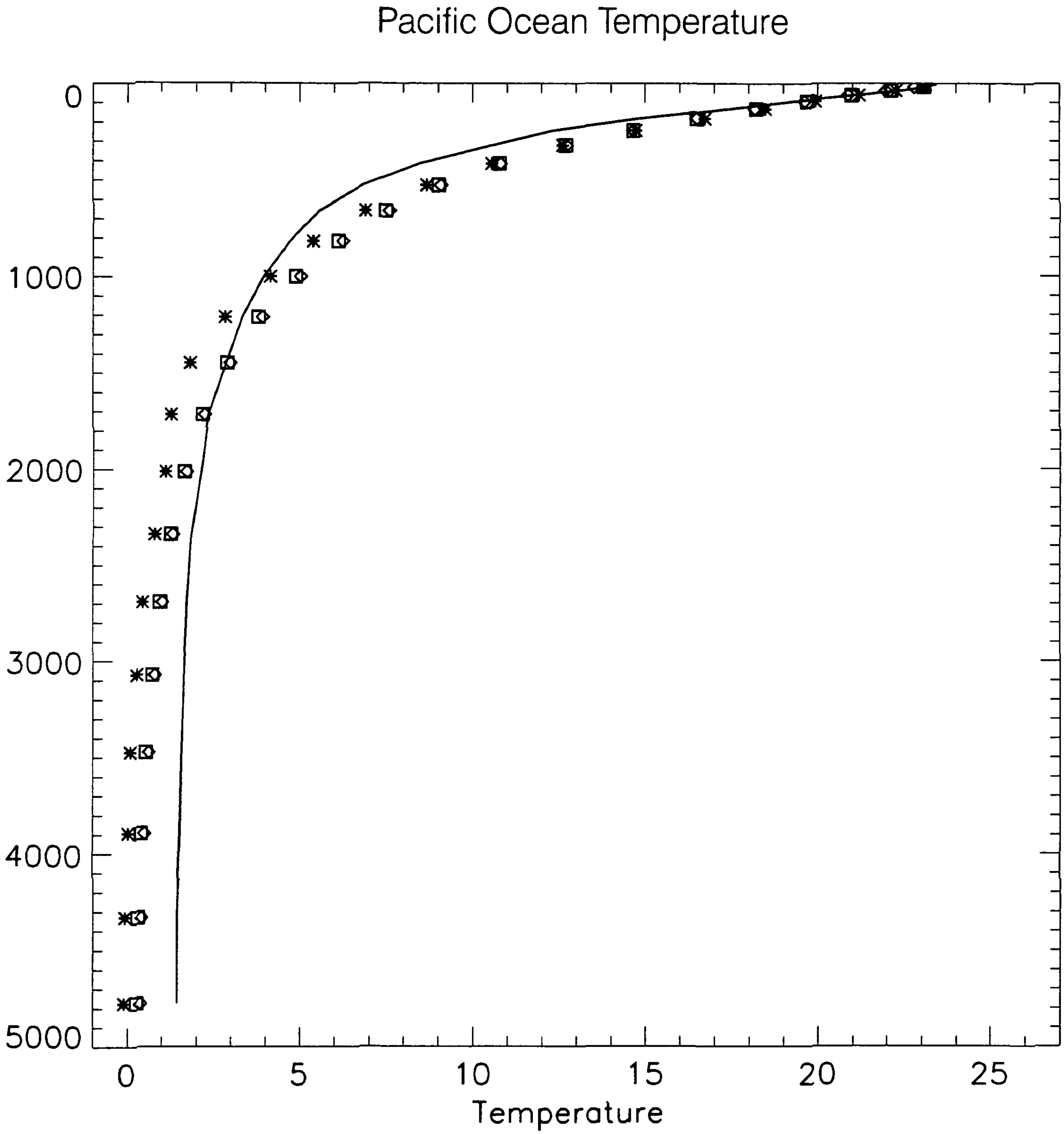

Pacific Ocean Temperature
0
1000
2000
3000
4000
5000
0
5
10
15
20
25
Temperature

Fig. 18

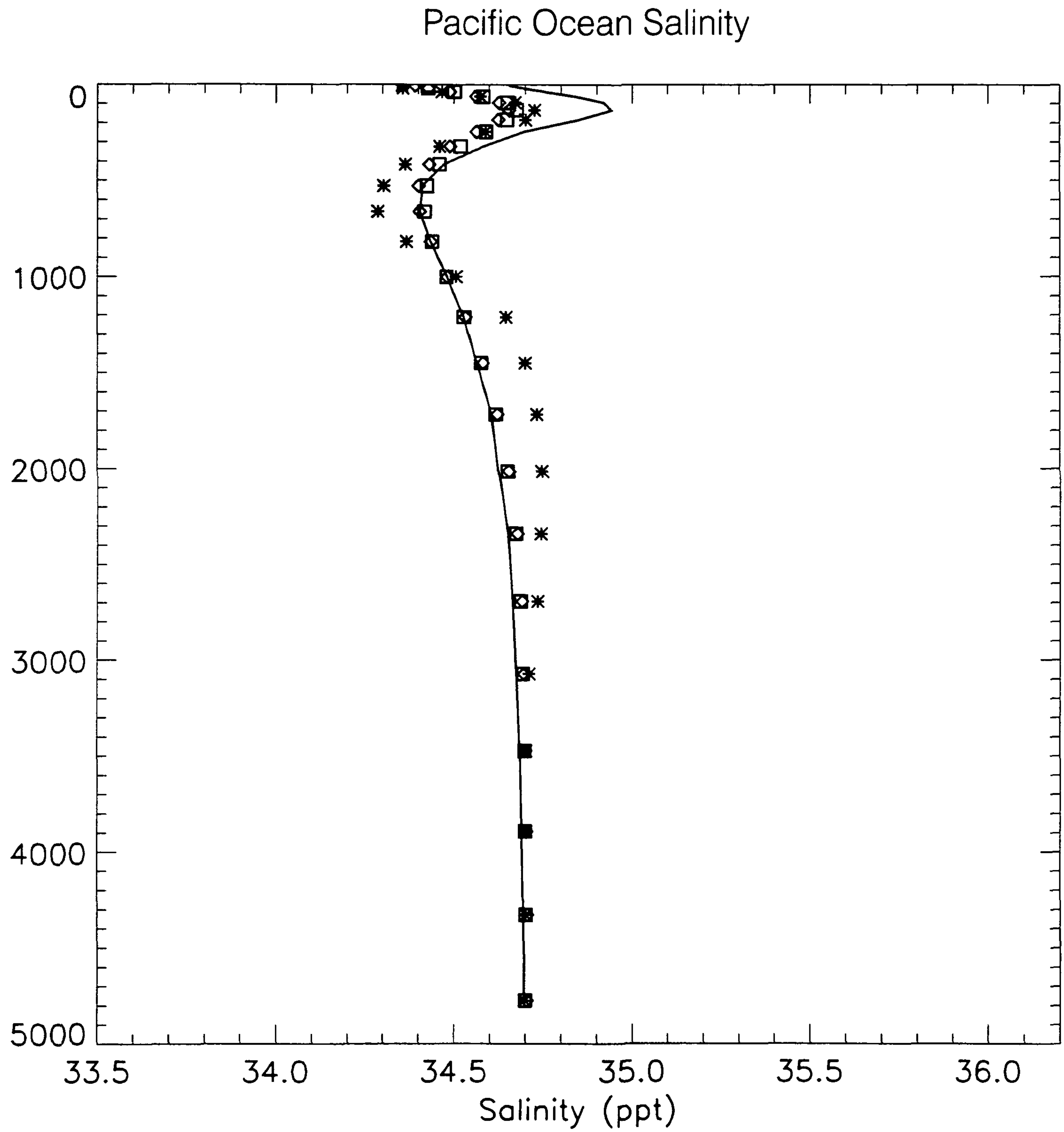

Pacific Ocean Salinity
0
1000
2000
3000
4000
5000
33.5
34.0
34.5
35.0
35.5
36.0
Salinity (ppt)

Fig. 19

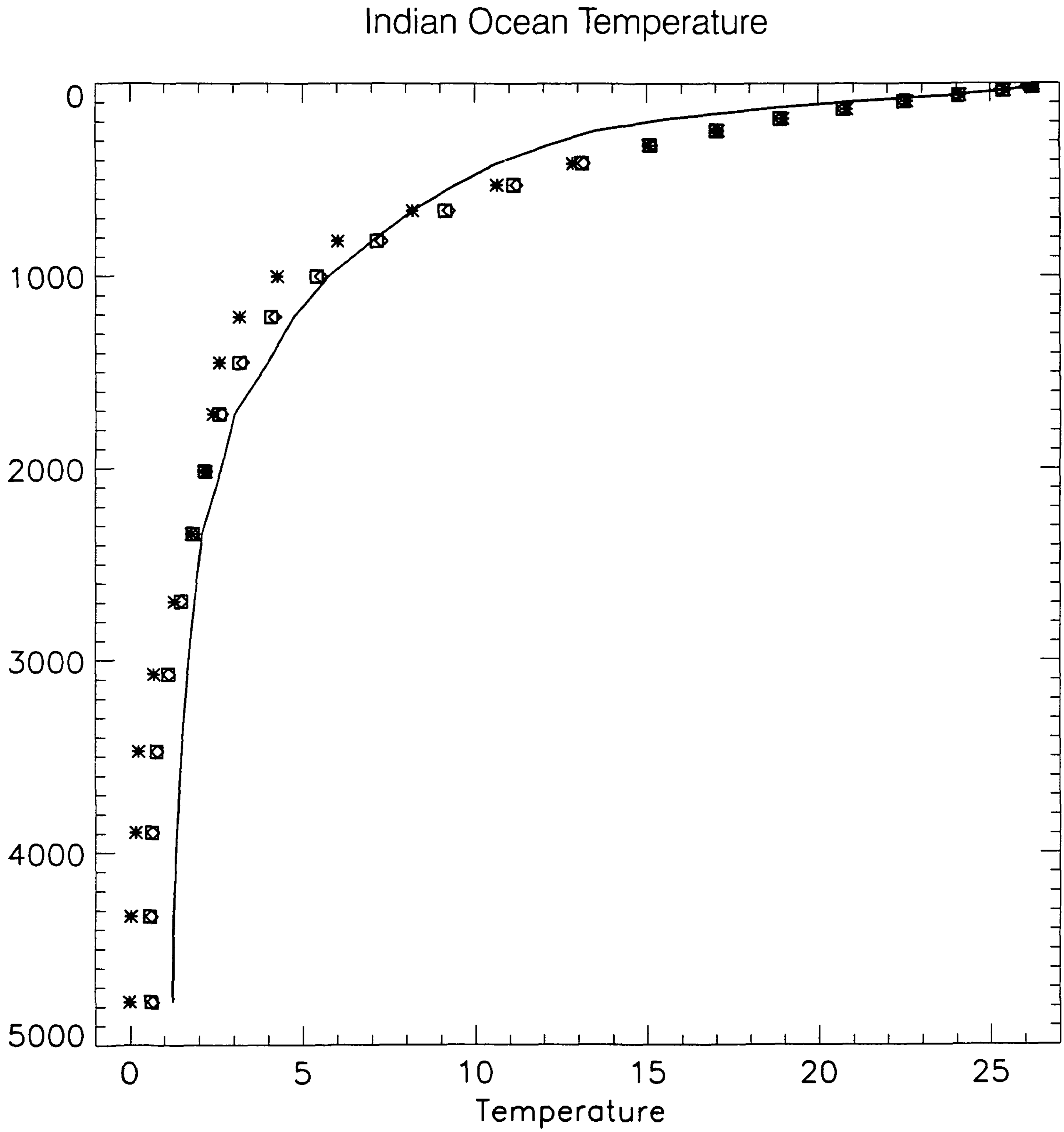

Indian Ocean Temperature
0
1000
2000
3000
4000
5000
0
5
10
15
20
25
Temperature

Fig. 20

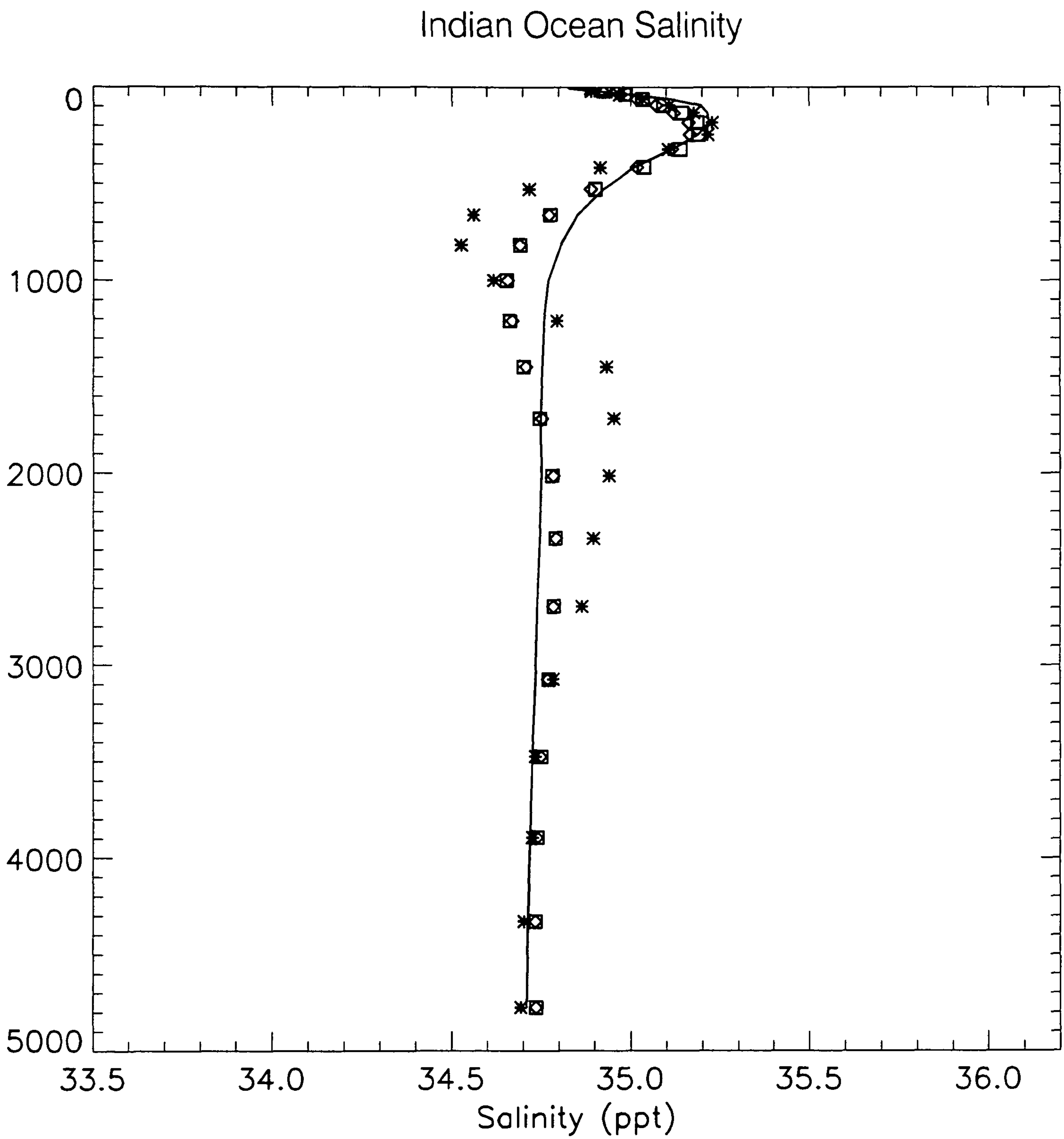
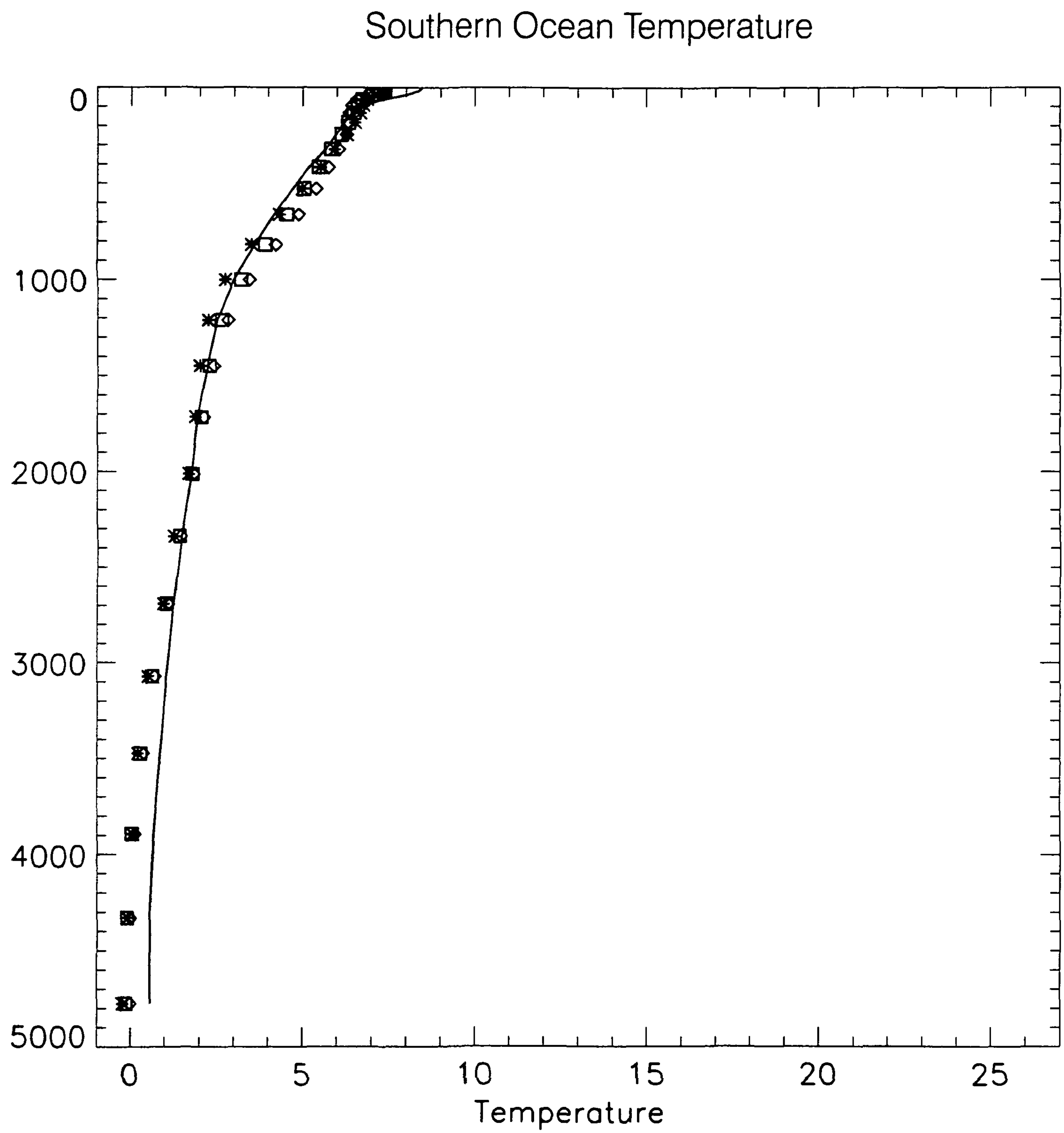

Indian Ocean Salinity
0
1000
2000
3000
4000
5000
33.5
34.0
34.5
35.0
35.5
36.0
Salinity (ppt)

Fig. 21



Southern Ocean Temperature
0
1000
2000
3000
4000
5000
0
5
10
15
20
25
Temperature

Fig. 22

Southern Ocean Salinity

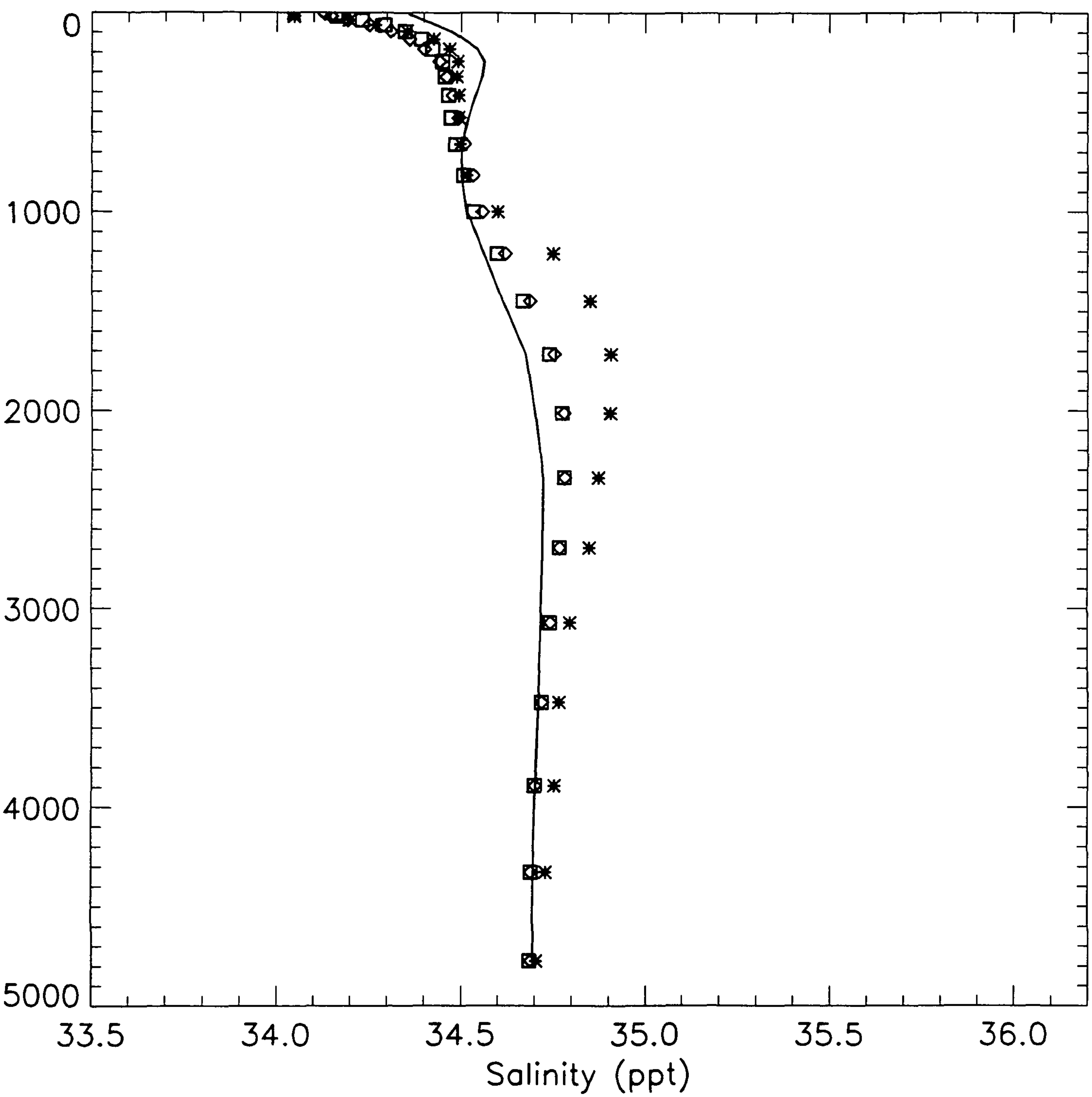
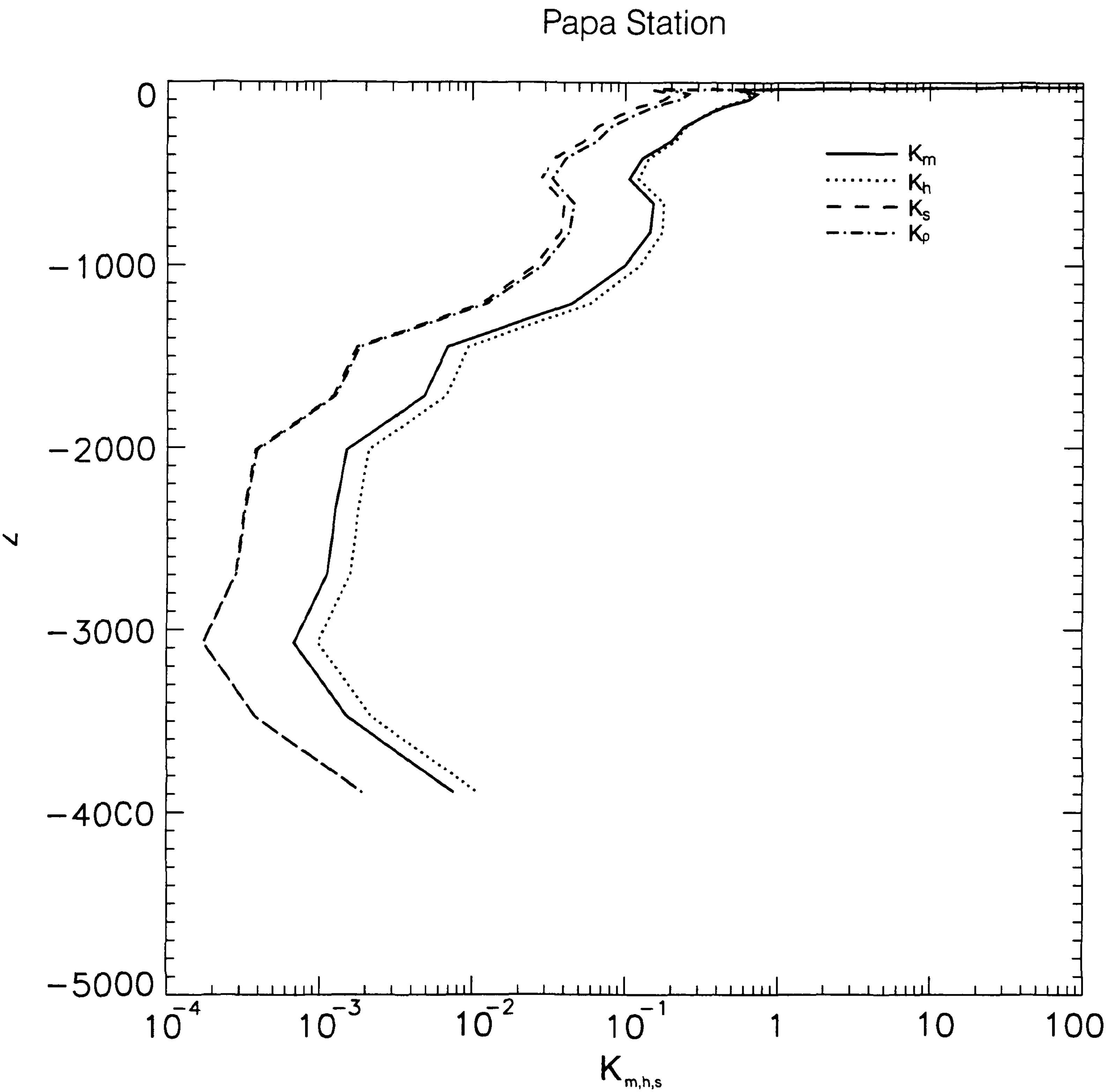


Fig. 23



Papa Station
0
−1000
−2000
−3000
−4000
−5000
z
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10
100
$K_{m,h,s}$
K_m
K_h
K_s
K_ρ

Fig. 24

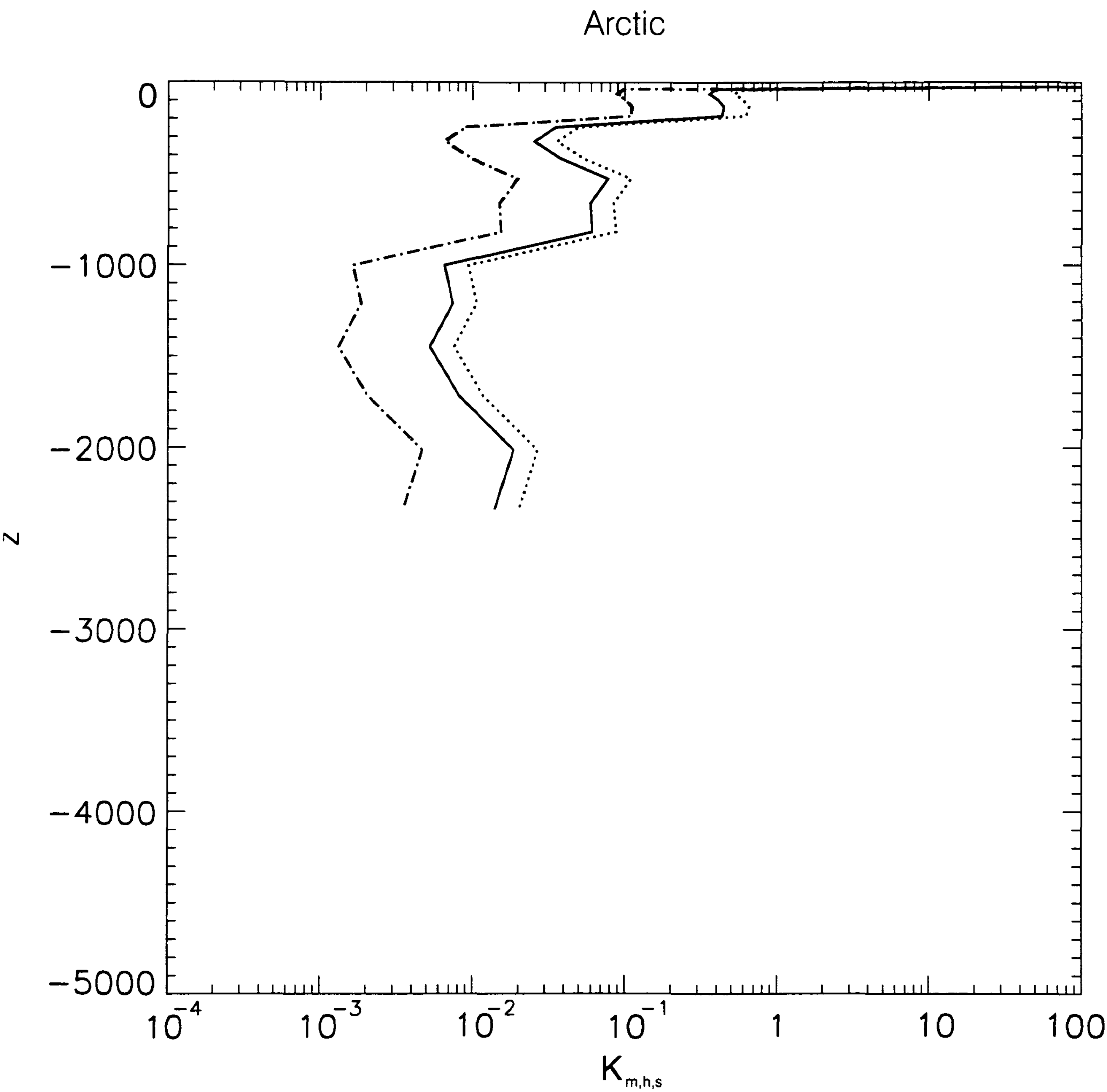
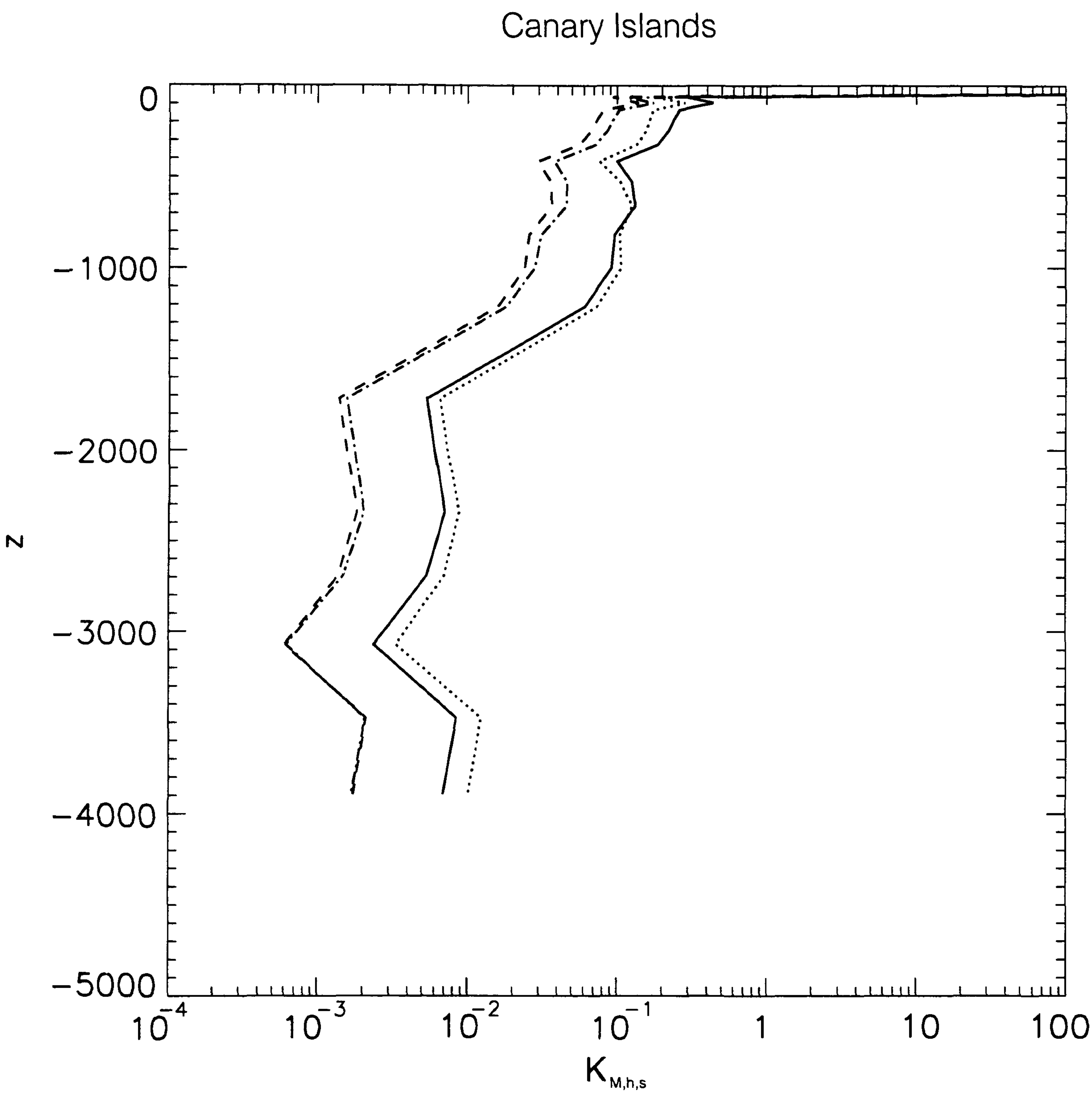

Arctic
0
−1000
−2000
−3000
−4000
−5000
z
10^-4
10^-3
10^-2
10^-1
1
10
100
K m,h,s

Fig. 25



Canary Islands
0
−1000
−2000
−3000
−4000
−5000
z
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10
100
$K_{M,h,s}$

Fig. 26

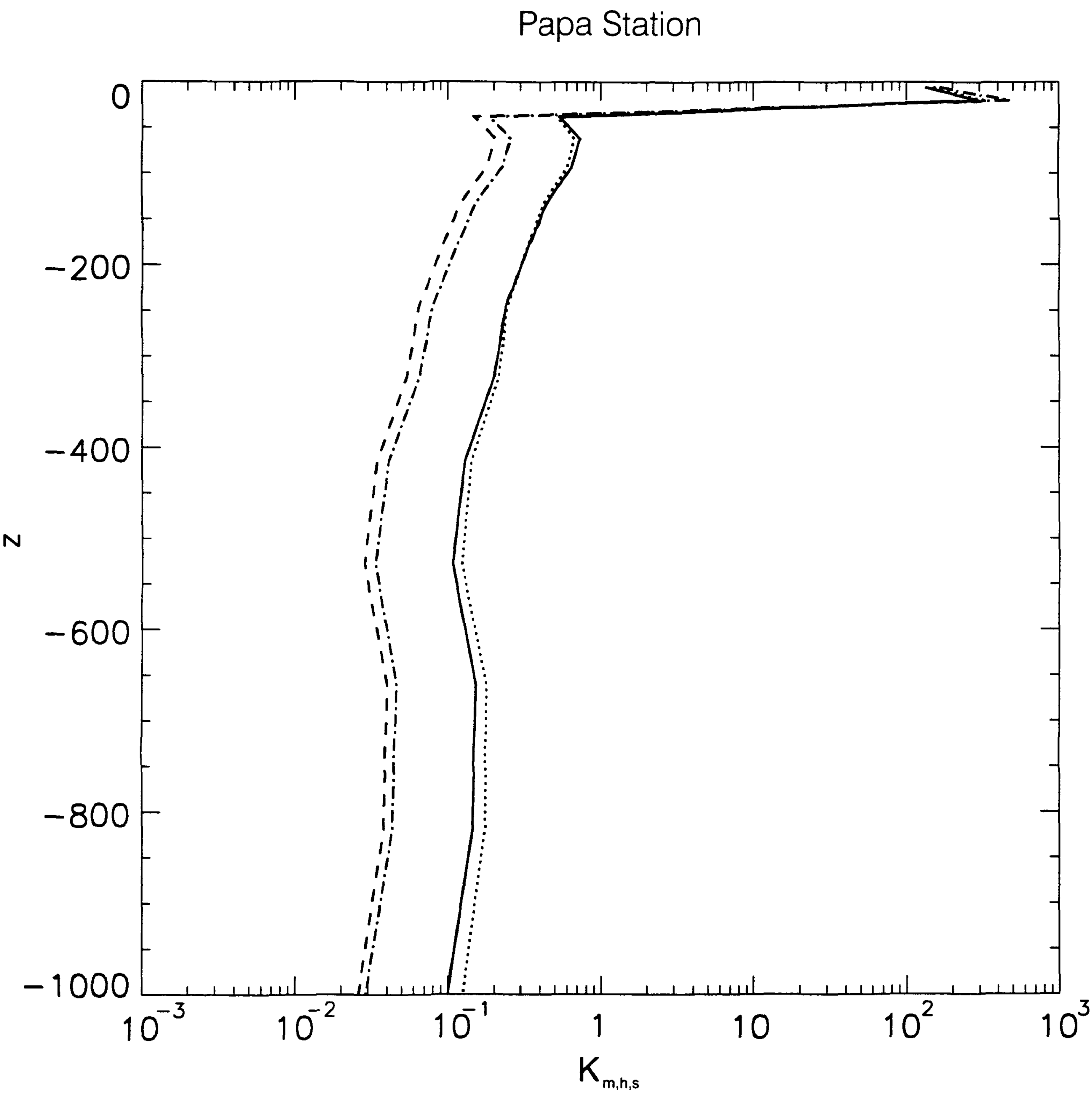

Papa Station
0
−200
−400
−600
−800
−1000
z
10^{-3}
10^{-2}
10^{-1}
1
10
10^{2}
10^{3}
$K_{m,h,s}$

Fig. 27

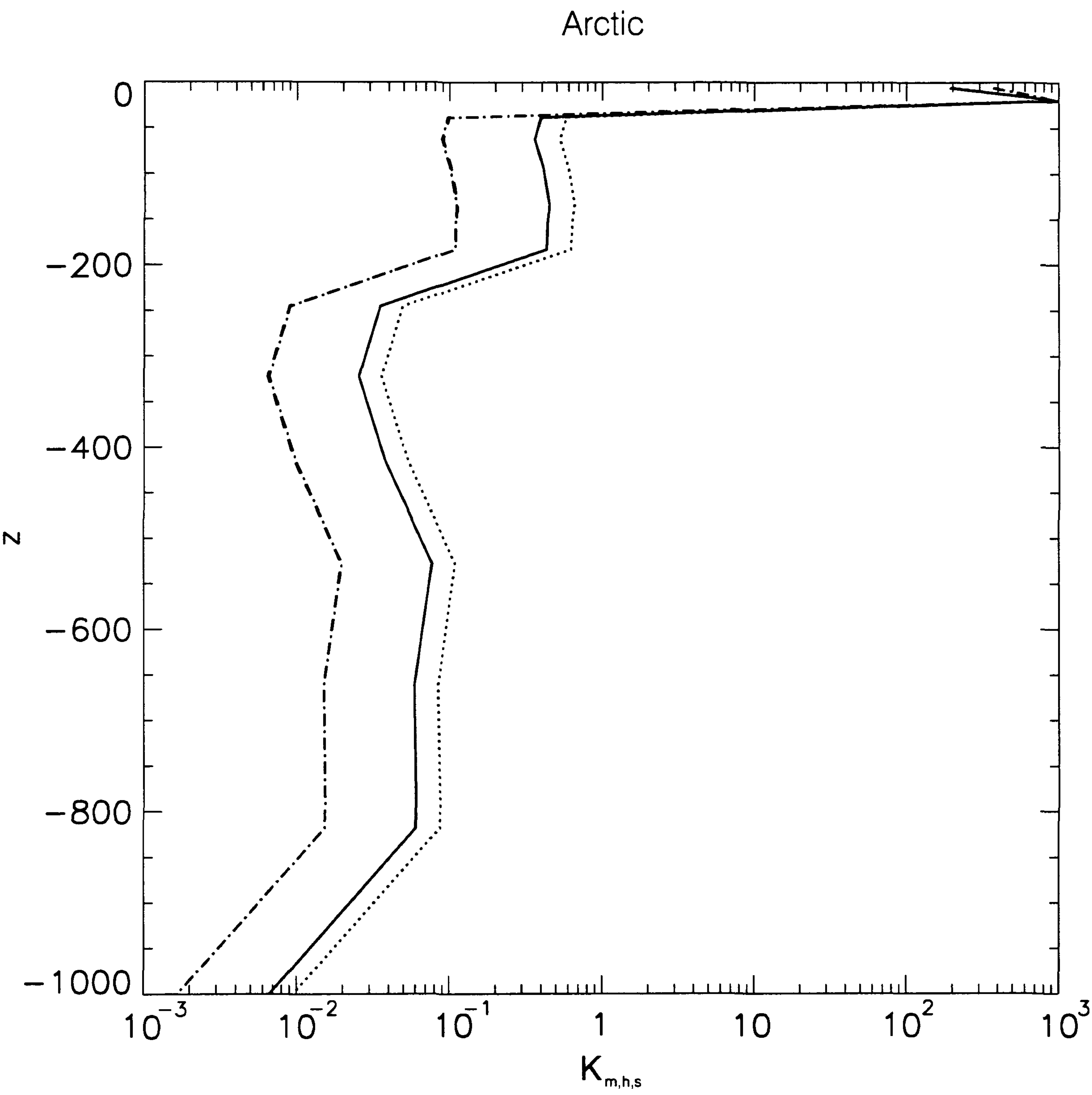
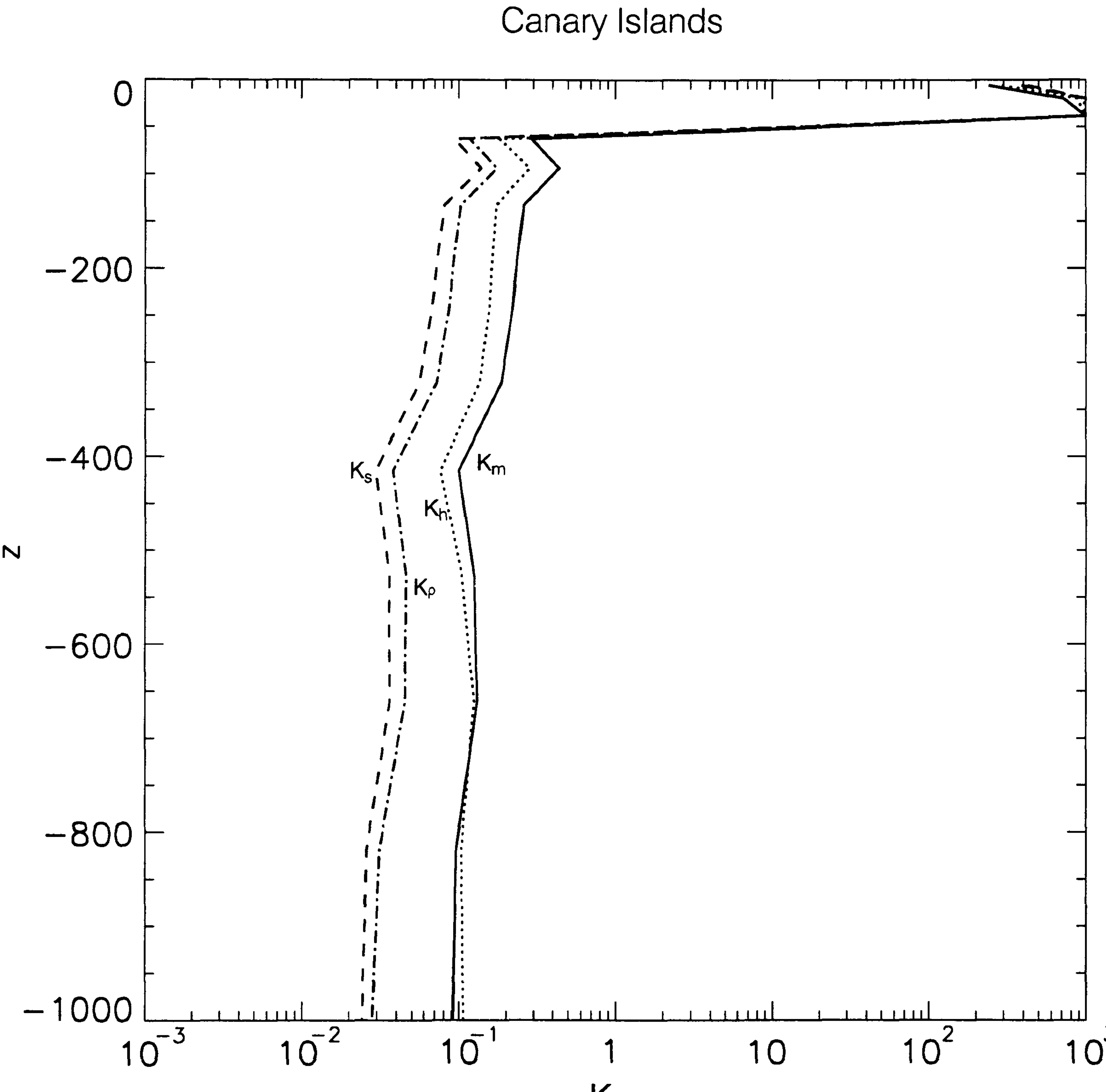


Fig. 28



Canary Islands
0
−200
−400
−600
−800
−1000
z
10^{-3}
10^{-2}
10^{-1}
1
10
10^2
10^3
$K_{m,h,s}$
K_s
K_m
K_h
K_ρ

Fig. 29

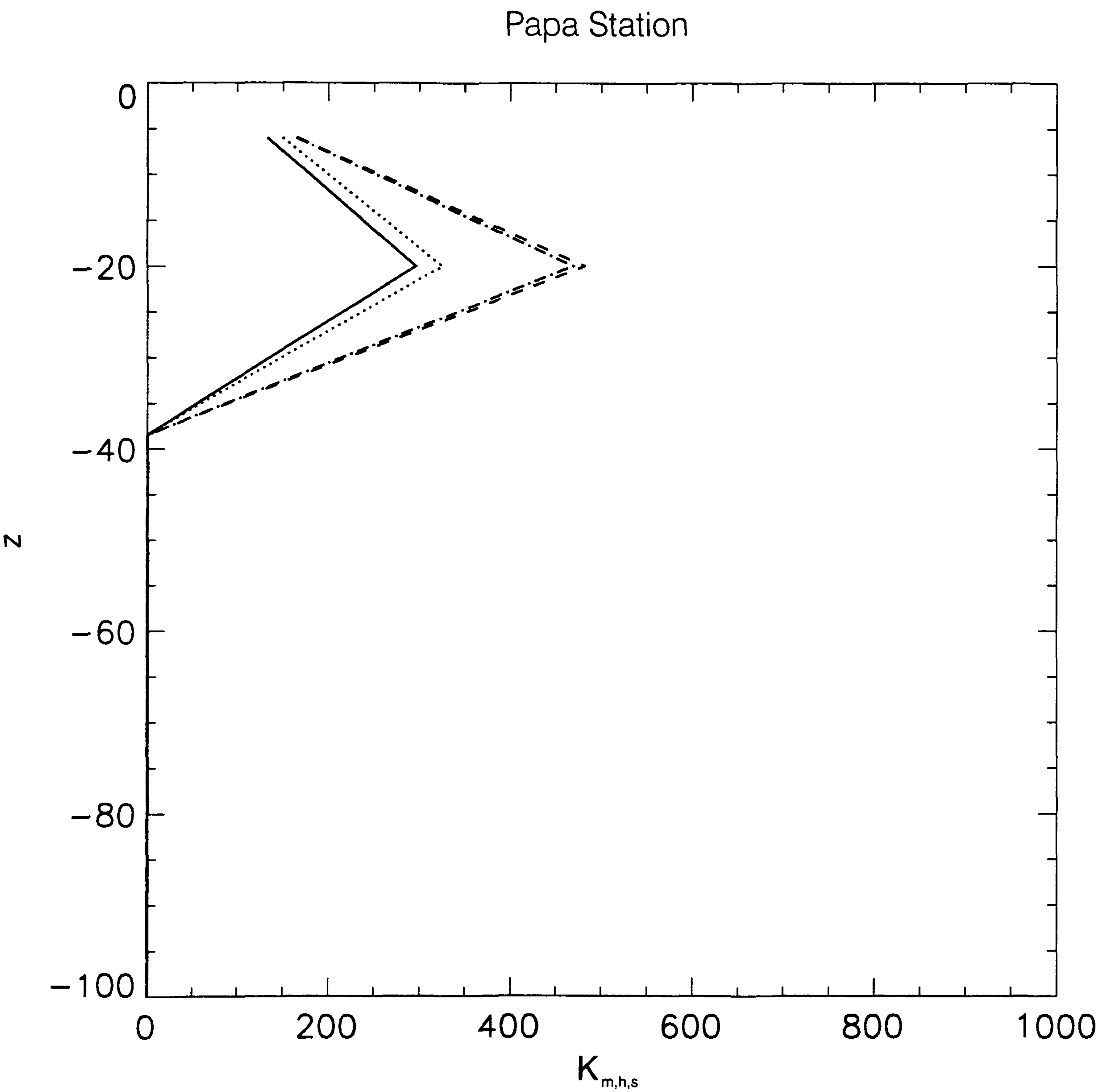


Fig. 30

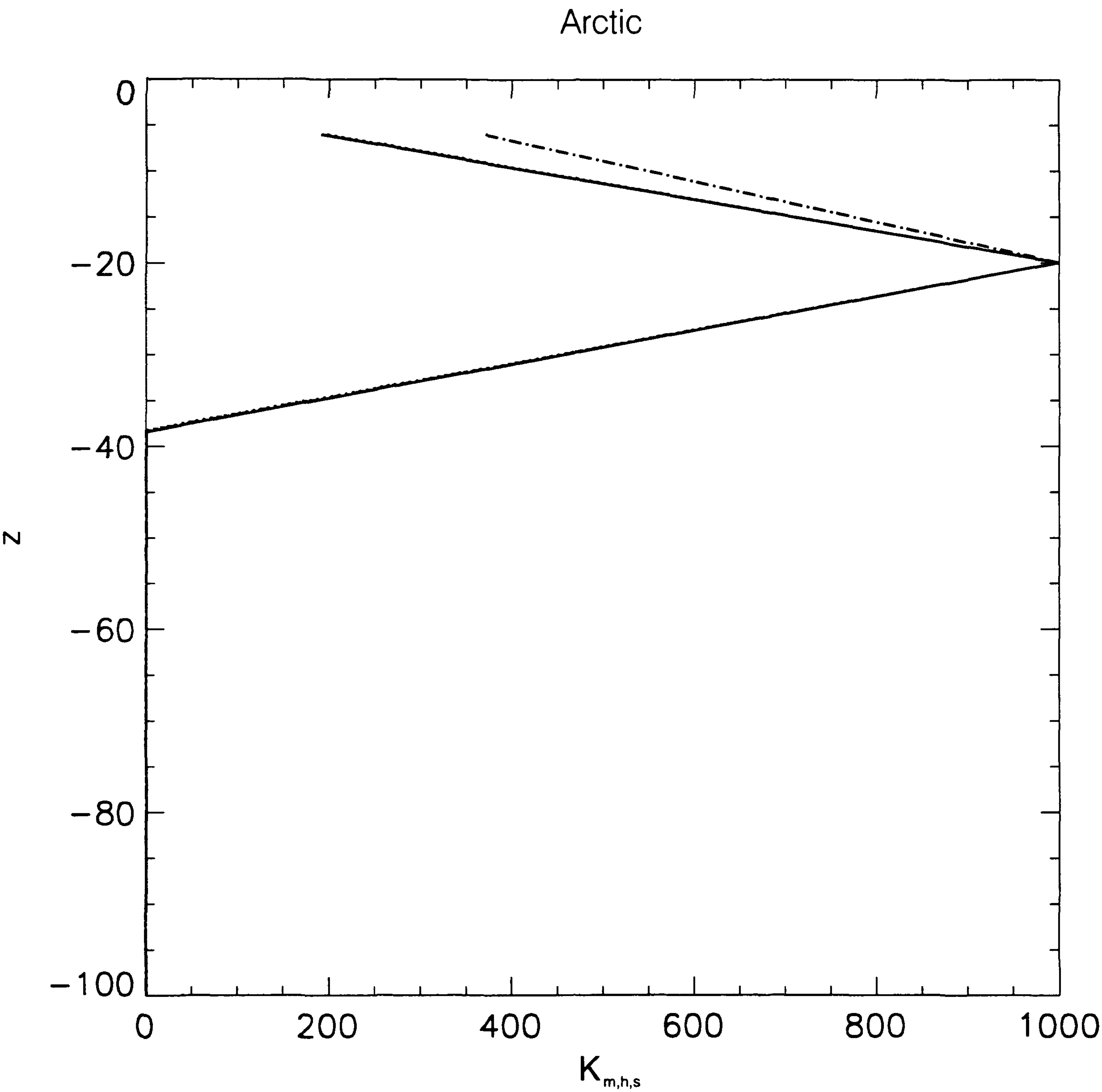


Fig. 31

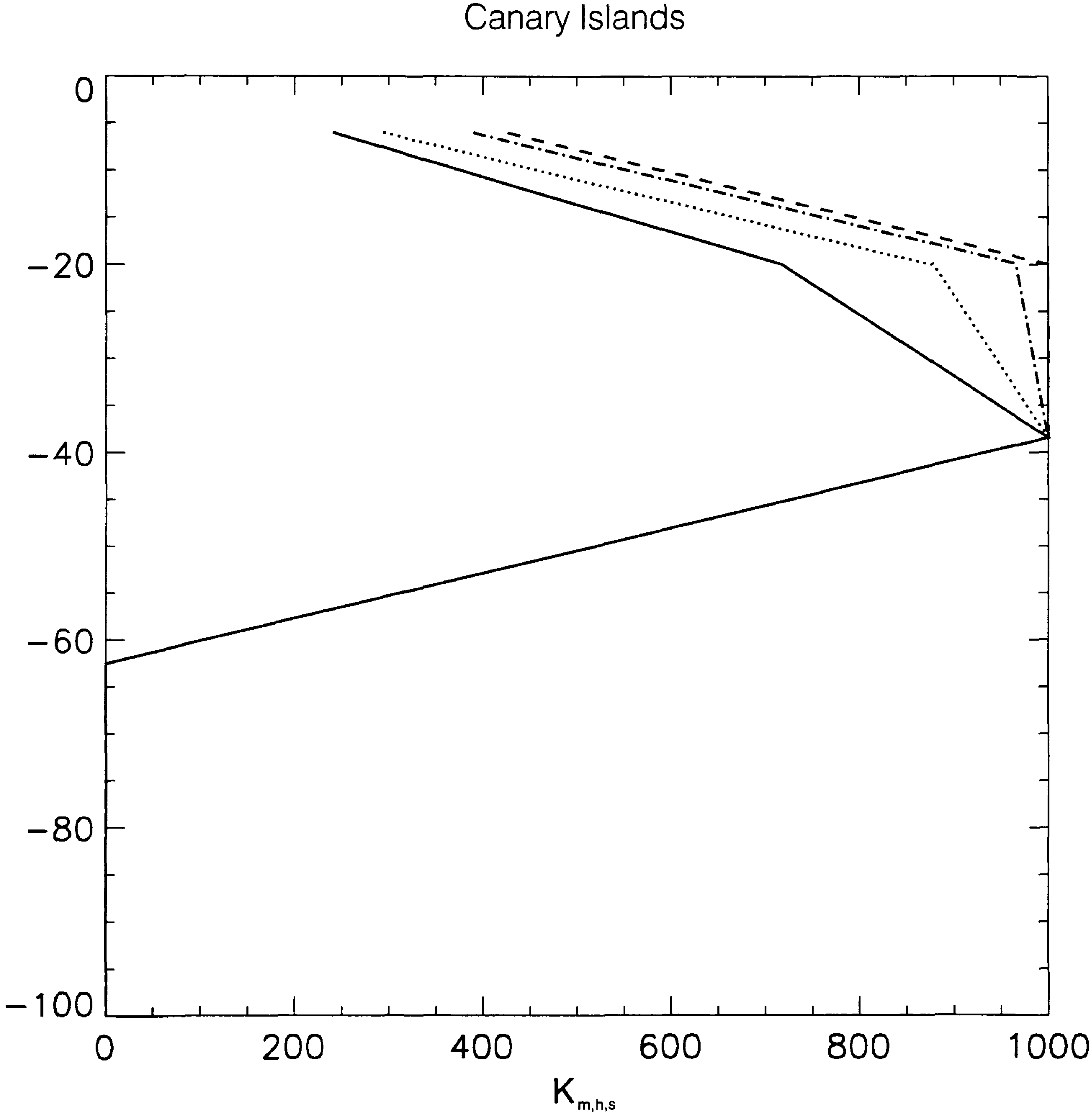
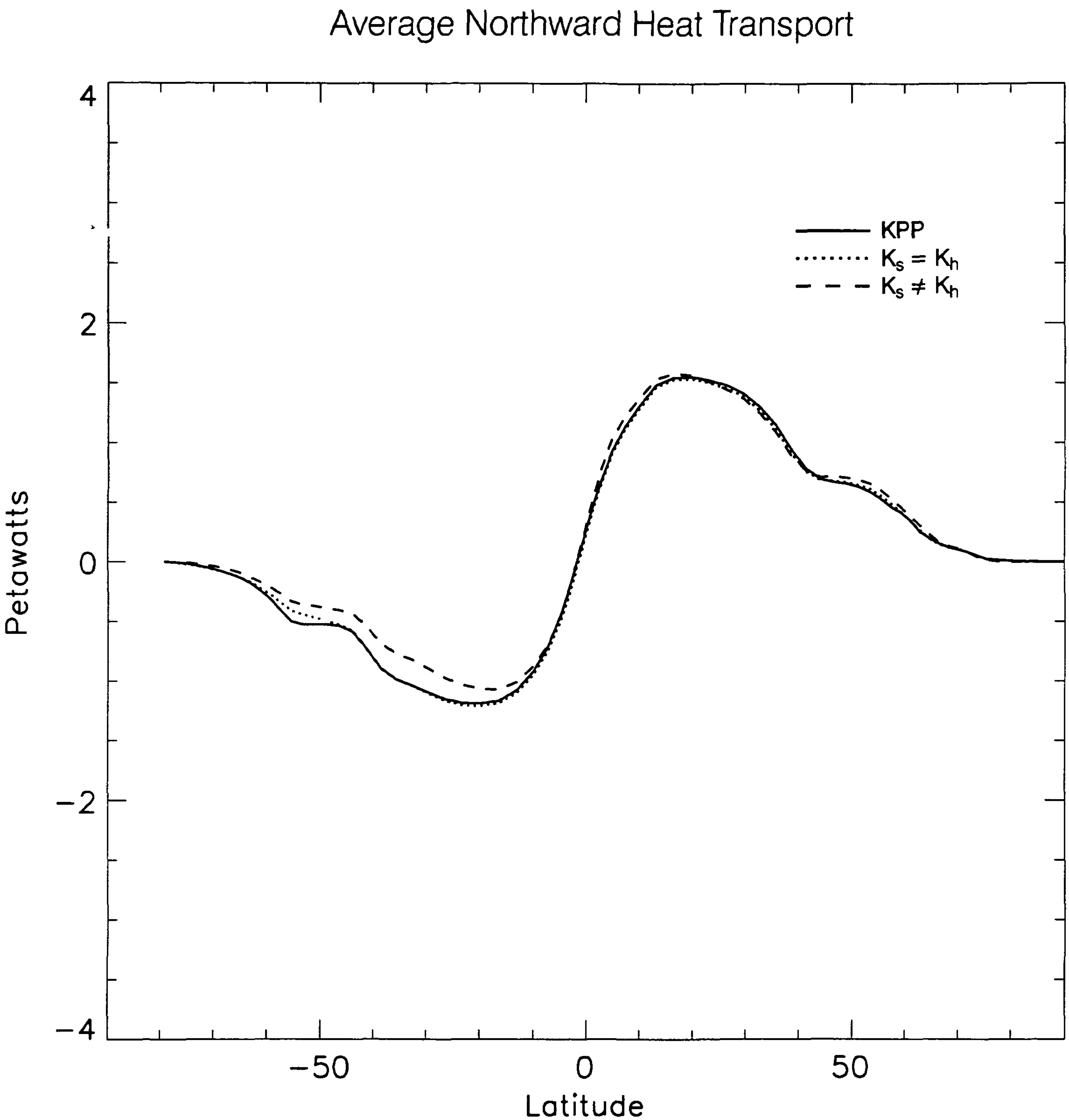


Fig. 32



Average Northward Heat Transport
KPP
$K_s = K_h$
$K_s \neq K_h$
4
2
0
−2
−4
Petawatts
−50
0
50
Latitude

Fig. 33